A GENERALIZATION OF COMPLETELY CONVEX FUNCTIONS AND RELATED PROBLEMS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "A GENERALIZATION OF COMPLETELY CONVEX FUNCTIONS AND RELATED PROBLEMS", submitted by David John Leeming in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

Following Schoenberg let $E_n^k = (\varepsilon_{ij})$ $(i=1,\dots,k;$ $j=0,1,\dots,n-1)$, $\varepsilon_{ij} = 0$ or 1, $\sum \varepsilon_{ij} = n$ be the incidence matrix of an interpolation problem of finding a polynomial $P(x)$ of degree $\leq n-1$ with prescribed values at k given real nodes $x_1 < \dots < x_k$ where $\varepsilon_{ij} = 1$ or 0 according as $P^{(j)}(x_i)$ is prescribed or not. The interpolation problem (equivalently E_n^k) is said to be real poised (order poised) if it has a unique solution for every choice of real distinct nodes $x_1,\dots,x_k$.

In Chapter I we introduce the (p,L) series for a given positive integer $p \geq 2$ as a generalization of Lidstone series by iterating the poised matrix $E_p^2 = \begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$. We show that if λ_1 is the smallest positive zero of the generalized sine function of order p ((2.8), p. 18) and if $f(z)$ is an entire function of exponential type $< \lambda_1$ then $f(z)$ has a (p,L) series representation for all z. This leads naturally to the study of W_p-convex and minimal W_p-convex functions which generalize the results of Widder on completely convex functions. Indeed, we show that $f(x)$ can be represented by an absolutely convergent (p,L) series if and only if it can be expressed as the difference of two minimal W_p-convex functions.

Chapter II deals with a three-point expansion called (p,L^*) series obtained by iterating the poised

matrix $E_p^3 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$, p even. Some results analogous to those of Chapter I are obtained and the class of W_p^*-convex functions is defined. However, in this case, the complete analogy to the results of Widder is lacking.

In Chapter III, we obtain the explicit form of the polynomial of $(0,n-1,n)$ interpolation for n given real nodes. Finally, we give some results on mean square convergence of the polynomials of $(0,n-1,n)$ interpolation on n^{th} roots of unity when f is a given analytic function in $|z| < 1$ and continuous on $|z| \leq 1$.

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A HISTORICAL SURVEY

1. Introduction. A problem of fundamental interest in classical analysis is to study the representation of an analytic function as the sum of a sequence of given functions $\{P_n(z)\}$. The school of J. M. Whittaker [34] considers the problem from a very general point of view, leading to the theory of "basic" series. Boas and Buck [7] point out the limitations of the theory of basic series in their monograph and discuss the expansion of analytic functions in series of polynomials defined by some generating relation. In general, interpolatory conditions can be translated into some suitable generating relation and, conversely, a generating relation can be interpreted in terms of some interpolatory conditions.

If we denote the set of interpolatory conditions by $L_n(f) = f^{(\alpha_n)}(a_n)$ $(n = 1,2,\ldots)$ for some prescribed numbers a_n and nonnegative integers α_n, then the interpolatory polynomials (or functions) are defined in such a way that $L_m(P_n) = \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta. Then, for any entire function, we may write a formal expansion

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} L_n(f)\, P_n(z)$$

and consider the following three problems of interpolation, as formulated by Evgrafov ([13], p. 251):

1. The problem of finding the class of functions for which the formal interpolation series (1) converges to $f(z)$.
2. The problem of finding a larger class of functions for which it is possible to construct uniquely the function $f(z)$ from the given $L_n(f)$. This, of course, includes the problem of defining methods to produce this construction.
3. The problem of finding the general form of functions for which $L_n(f)$ has prescribed values (e.g., all $L_n(f)$ are equal to zero).

Such interpolation series can provide a means of penetration into various properties of entire functions as was brought out in a long survey article by Evgrafov. There, he has solved a problem first posed in 1937 on Abel-Gontcharoff interpolation where $L_n(f) = f^{(n)}(\lambda_n)$ with $\lambda_n = n^{1/p}$. The methods devised by Evgrafov for solution of this problem have much wider significance than merely the question of interpolation.

A. O. Gel'fond and A. I. Markuševič brought out the intimate connection between the problem of convergence of interpolation series, the problem of whether or not a given system of functions is complete and whether or not it forms a basis in some space of analytic functions. For an extensive bibliography of the Russian contributions in this area, see Evgrafov [13].

2. Poised Problems of Polynomial Interpolation. Following Schoenberg [24], we shall use an incidence matrix $E_n^k = (\varepsilon_{ij})$ $(i = 1,2,\ldots,k;\ j = 0,1,\ldots,n-1)$, $\varepsilon_{ij} = 0$ or 1 and $\sum_{i,j} \varepsilon_{ij} = n$ to describe the interpolation problem of finding a polynomial of degree $\le n-1$ with prescribed values and derivatives at k given points. Also, $\varepsilon_{ij} = 1$ (or 0) according as there is (or is not) a prescribed derivative of order j at the i^{th} node. Thus Lidstone interpolation is described by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and Abel-Gontcharoff interpolation by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and so on. We use the convention that no row is composed entirely of zeros. If $m_p = \sum_{i=1}^{k} \varepsilon_{ip}$ and $M_p = \sum_{j=0}^{p} m_j$, $p = 0,1,\ldots,n-1$, $M_{n-1} = n$, then E_n^k is said to have Polya property if $M_p \ge p+1$, for all p , and is said to have strong Polya property if $M_p > p+1$, for all p . An interpolation problem is said to be poised (or real poised, or n-poised) if the problem is uniquely solvable for given members $y_i^{(j)}$ for all choices of real and distinct nodes $x_1 < x_2 < \ldots < x_k$. Thus, Lagrange, Hermite, Lidstone interpolation with its generalizations by Poritsky [21], and Abel-Gontcharoff interpolation (see e.g. [34]) are all similar in a sense, since they are all poised problems of interpolation.

Polya showed that a necessary and sufficient condition for a two-point interpolation problem to be poised is that $M_p \geq p+1$, $p = 0,1,\ldots,n-1$. For the k-point interpolation problem $(k > 2)$, simple necessary and sufficient conditions are unknown. Sufficient conditions have recently been given by Atkinson and Sharma [1], and Sharma and Prasad [27], but Lorentz and Zeller (in a paper to be published) use a simple example to show that these conditions are not necessary.

3. Non-poised Problems of Polynomial Interpolation.

J. Suranyi and P. Turán were the first to undertake a study of non-poised interpolation problems in their paper on (0,2) interpolation. This notation is used to indicate that the values of the function and its second derivatives are prescribed at some n given points. They showed that if the n nodes are the zeros of $(1-x^2)\,P'_{n-1}(x)$, where $P_{n-1}(x)$ is the Legendre polynomial of degree $n-1$, then, for n even, the polynomials of (0,2) interpolation exist and are unique; however, this is not so for n odd.

However, it does not appear to be a simple problem, to find the explicit forms for the interpolatory polynomials even if we know that the interpolation problem is uniquely solvable (or poised) for any given real nodes. Thus, it follows from the result of Atkinson and Sharma [1], that

(0,2,3) interpolation is poised, but the formulae for these polynomials with n given nodes are unknown.

4. <u>Interpolation by Entire Functions</u>. If there are infinitely many prescribed interpolatory conditions, we may consider the problem from the point of view of an infinite system of linear equations in infinitely many unknowns. Guichard (see Davis [12], p. 96) showed that it is possible to find an entire function f satisfying infinitely many interpolation conditions on f of the form $f^{(\alpha_n)}(z_n) = a_n$ $(n = 1,2,\dots)$, provided $\lim_{n\to\infty} z_n = \infty$. Polya [19] showed that no entire function may exist satisfying infinitely many interpolatory conditions, if the sequence of interpolation points is bounded. However, Vermes [30] has determined some sufficient conditions for the existence of an entire function satisfying infinitely many interpolatory conditions on two nodes. On the other hand, if the interpolation conditions are periodic with period p (see Polya [19]), then the interpolation problem has a unique solution, provided the first p prescribed conditions yield a poised interpolation problem.

Much of the present work was motivated by the results of Polya [19] and Schoenberg [24] on Hermite-Birkhoff interpolation along with some of its recent extensions ([1], [27]). Sharma and Prasad [27] have shown that if two interpolation problems defined by incidence matrices F_p^k and G_r^k are

poised, then the "sum" of these matrices E_n^k $(n = p+r)$ also defines a poised interpolation problem. Thus it is possible to consider an infinite interpolation problem with periodic interpolatory conditions from another point of view. Here we begin with a matrix E_n^k defining a poised interpolation problem and consider the infinite periodic interpolation problem produced by successively iterating the matrix E_n^k. It is clear that in this way the interpolatory conditions will be periodic of period n. Also, Schoenberg [25] has considered the infinite interpolation problem defined by

$E = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \ldots \\ 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 1 & \ldots \end{pmatrix}$ with nodes -1, 0 and 1, which is an analogue of Lidstone's two-point expansion, but no longer has periodic interpolatory conditions. The expansion formula obtained in this case is

$$f(x) = f(0)A_0(x) + \sum_{n=0}^{\infty} f^{(2n)}(1)B_n(x) + \sum_{n=0}^{\infty} f^{(2n)}(-1)B_n(-x) ,$$

where $A_0(x)$, $B_n(x)$ $(n = 0,1,\ldots)$ are entire functions of exponential type $\frac{\pi}{2}$. There are several open questions concerning such an expansion. For example, does the expansion converge to the function for all entire functions $f(x)$ of exponential type $< \pi$? The example $f(x) = \sin \pi x$ shows that the bound cannot be larger than π. A three-point interpolation problem analogous to that of Schoenberg [25] can be obtained by considering the matrix

$$E^* = \begin{pmatrix} \underbrace{\begin{matrix} 1 & 1 & .. & 1 & 0 \\ 1 & 0 & .. & 0 & 0 \\ 1 & 0 & .. & 0 & 0 \end{matrix}}_{p \text{ columns}} & \underbrace{\begin{matrix} 1 & 1 & .. & 1 & 0 \\ 0 & 0 & .. & 0 & 0 \\ 1 & 0 & .. & 0 & 0 \end{matrix}}_{p \text{ columns}} & \begin{matrix} \ldots \\ \ldots \\ \ldots \end{matrix} \end{pmatrix} .$$

5. <u>Absolutely and Completely Monotonic Functions</u>.

S. Bernstein [2] introduced the term <u>absolutely monotonic</u> to describe functions which are nonnegative on some interval $a < x < b$ and have nonnegative derivatives of all orders on that interval. For example, e^x is absolutely monotonic on any interval. He showed that functions absolutely monotonic for $-\infty < x < 0$ are necessarily analytic and have the representation $f(x) = \int_0^\infty e^{xt} \, d\alpha(t) < \infty$, $(\alpha(t)\uparrow, -\infty < x < 0)$. Widder [36] obtained this representation independently without knowledge of Bernstein's result. Bernstein also proved the following:

<u>Theorem</u> (Bernstein, [2]). <u>A necessary and sufficient condition that it should be possible to expand the function</u> $f(x)$ <u>in a series of powers of</u> $(x-a)$ <u>convergent for</u> $a \leq x < b$ <u>is that</u> $f(x)$ <u>should be the difference of two functions absolutely monotonic in</u> $a \leq x < b$.

Since the Taylor series expansion of a function can be considered as a one-point interpolation problem, the above theorem provides the motivation for the study of the relationship between an infinite interpolation problem and a particular class of functions with derivatives of all orders on an interval. Later, Widder [38] showed the connection between Lidstone interpolation and the class of completely convex functions of §6.

A function $f(x)$ is said to be completely monotonic for $a < x < b$ if $f(-x)$ is absolutely monotonic for $-b < x < -a$. Functions completely monotonic for $0 < x < \infty$ have the representation $f(z) = \int_0^\infty e^{-xt} d\alpha(t) < \infty$, ($\alpha(t)\uparrow$, $0 < x < \infty$). Additional results on absolutely and completely monotonic functions, along with the results mentioned here, are given in [39] (Chapter IV).

6. Regularly Monotonic Functions, Completely Convex Functions and their Generalizations. Bernstein [3] also considered the class of regularly monotonic functions; i.e., functions each of whose derivatives are of constant sign in an interval. He classified these functions in terms of "typical" numbers $\lambda_1, \lambda_2, \ldots$ indicating the number of successive derivatives maintaining the same constant sign. In this way, the derivatives are put into "blocks" such that $f^{(n)}(x)$ and $f^{(n+1)}(x)$ belong to different blocks if and only if $f^{(n)}(x) f^{(n+2)}(x) < 0$. Bernstein found the relationship between the lengths of the blocks and the analytic nature of the function. This relationship is aptly described in the words of Boas and Polya ([8], p. 406): "Roughly stated, the analytic nature of $f(x)$ is simpler if the blocks are shorter."

A particular class of regularly monotonic functions with the property that $\lambda_n = 1$ $(n = 1,2,\ldots)$ is called cyclically monotonic and was studied in some detail by

Bernstein [4]. He proved that if $f(x)$ is cyclically monotonic on $[0,b]$ then it must necessarily be entire of exponential type not exceeding $\frac{2}{b}$. Also, he proved the following:

<u>Theorem</u> (Bernstein, [4]). <u>A function</u> $f(x)$ <u>is entire of exponential type at most</u> b <u>if and only if it can be represented on any interval of length less than</u> $\frac{\pi}{2b}$ <u>as the difference of two cyclically monotonic functions, but not so represented on some interval of greater length</u>.

Widder introduced the term <u>completely convex</u> to describe functions satisfying the inequalities $(-1)^k f^{(2k)}(x) \geq 0, \quad (k = 0,1,\ldots)$ on an interval $a \leq x \leq b$. Unlike the absolutely monotonic, completely monotonic, or cyclically monotonic functions for which every derivative has a prescribed sign, there are no conditions on any derivative of odd order of a completely convex function. It is easily seen that if a function $f(x)$ is cyclically monotonic on an interval, then either $f(x)$ or $-f(x)$ is completely convex on that interval. However, the function $\sin \pi x$ which is completely convex on $[0,1]$ is not cyclically monotonic on $[0,1]$.

Widder [38] showed that a function which is completely convex in an interval (a,b) is necessarily entire of finite exponential type. He also showed [38] that each term in the Lidstone series expansion of a

completely convex function is nonnegative.

Following Widder, we say that a function $f(x)$ is <u>minimal completely convex</u> in (a,b) if $f(x)$ is completely convex there and if $f(x) - \varepsilon \sin \pi x$ is not completely convex in that interval for any $\varepsilon > 0$. Then we have the following:

<u>Theorem</u> (Widder, [38]). <u>A necessary and sufficient condition that $f(x)$ can be represented by an absolutely convergent Lidstone series is that it should be the difference of two minimal completely convex functions on</u> $0 \le x \le 1$.

The first paper of Widder [37] on completely convex functions (in 1940) generated considerable mathematical activity, producing generalizations in several directions, all of which were published in 1941 and 1942.

Boas and Polya [8] gave some general results on functions with certain prescribed derivatives which do not change sign on the interval $[-1,1]$. Since they are so closely related to the work contained herein, we state the main results here. The first theorem contains both Bernstein's results on regularly monotonic functions and Widder's results on completely convex functions.

<u>Theorem 1</u> (Boas and Polya, [8]). <u>Let</u> $\{n_k\}\uparrow$ <u>and</u> $\{q_k\}$ <u>be sequences of positive integers. Let</u> $f(x)$ <u>be real valued and of class</u> $C^{\infty}[-1,1]$. <u>For</u> $k = 1,2,\ldots$, <u>let</u> $f^{(n_k)}(x)$ <u>and</u> $f^{(n_k+2q_k)}(x)$ <u>not change sign in</u> $[-1,1]$,

<u>and let</u> $f^{(n_k)}(x) f^{(n_k+2q_k)}(x) \leq 0$. <u>Then if</u> (i) $n_k - n_{k-1} = O(1)$ <u>and</u> (ii) $q_k = O(1)$, $f(x)$ <u>coincides with an entire function of growth not exceeding order one and finite type</u>.

The second theorem gives a direct generalization of Widder's result on completely convex functions, which is obtained by setting $n_1 = 2$ and $q_k = 1$ $(k = 1,2,\ldots)$.

<u>Theorem 2</u> (Boas and Polya, [8]). <u>Let</u> $\{n_k\}\uparrow$ <u>be a sequence of even integers</u>. <u>Let</u> $f(x)$ <u>be real valued and of class</u> $C^\infty[-1,1]$, <u>and let</u> $(-1)^k f^{(n_k)}(x) \geq 0$, $(k = 1,2,\ldots)$. <u>Then if</u> $n_k - n_{k-1} = O(1)$, $f(x)$ <u>coincides with an entire function of growth not exceeding order one and finite type</u>.

Further generalizations of these results have been given by Wiener and Polya, Szegö, Hille and Schaeffer [20]. In order to give a brief survey of these results, we let N_n denote the number of changes of sign of $f^{(n)}(x)$ in an interval I . Wiener and Polya showed that if $f(x)$ is a 2π-periodic function and if $N_n \leq 2m$ for any n , then f is a trigonometric polynomial of order not exceeding m . Szegö proved that if $f(x)$ is periodic and $N_n < \frac{2n}{\log n}$ $(n \to \infty)$ then $f(x)$ is entire, and if $N_n = O(1)$ then $f(x)$ is of exponential type. Schaeffer generalized the result of Wiener and Polya by showing that if $N_n \leq M$ $(n = 1,2,\ldots)$ for some fixed M then $f(x)$ is analytic in I . Hille proved that if $N_n^* = O(1)$ on the interval

$[-1,1]$ then $f(x)$ is a polynomial, where N_n^* denotes the number of changes of sign of

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}\right]^n f(x) \quad .$$

7. <u>Summary</u>. In each of the three chapters of this thesis we are concerned first with determining a set of polynomials satisfying certain interpolatory conditions, which we call the fundamental polynomials of the particular interpolation problem in question. In Chapter I, the infinite interpolation problem is defined by successively iterating the incidence matrix $E_p^2 = \begin{pmatrix} 1 & 1 & 1 & & 1 & 0 \\ 1 & 0 & 0 & .. & 0 & 0 \end{pmatrix}$ with nodes 0 and 1 . In Chapter II, the infinite interpolation problem is defined by iterating the incidence matrix $E_p^3 = \begin{pmatrix} 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 1 & ... & 1 & 0 \\ 1 & 0 & 0 & & 0 & 0 \end{pmatrix}$ (p a positive even integer) with nodes -1, 0 and 1 . In Chapter III we define the fundamental polynomials of $(0, n-1, n)$ interpolation. By obtaining the explicit form of these polynomials, we show existence for every choice of n real nodes. However, as a simple example shows, for some choices of complex nodes the $(0, n-1, n)$ interpolation polynomials do not exist.

Once these fundamental polynomials are known we consider first the problem of expansion of an entire (or analytic) function in terms of such polynomials. To do this, we introduce the term (p,L) series in Chapter I, and (p,L^*) series in Chapter II. In Chapter III, we give some

results on uniform and least squares convergence for functions analytic in the unit disk with certain prescribed conditions on the boundary. The convergence theorems turn out to be similar to known convergence theorems ([26] and [28]) for Lagrange interpolation polynomials, as one would expect.

The second problem considered in Chapters I and II was motivated by Widder's work on Lidstone series and completely convex functions [38]. From this elegant result, we are led to consider the problem of defining a suitable class of functions corresponding to a given interpolation problem. Such considerations lead to the definition of W_p-convex functions and the generalization of Widder's result in Chapter I.

In an attempt to obtain results analogous to those of Chapter I, we define the class of W_p^*-convex functions in Chapter II. Some sufficient conditions are given for the representation of a function by a (p,L^*) series. However, as is pointed out in Chapter II, we are unable to obtain necessary conditions for representation of a function by an absolutely convergent (p,L^*) series. Some results are also given in Chapter II for the special case $p = 2$, relating a set of interpolation polynomials of the $(2,L^*)$ series to the Euler Polynomials.

Finally, in order to bring out the correspondence between certain classes of functions and the interpolation

problems which generate them, we give the following table:

Incidence matrix	Class of Functions	References
(1)	absolutely monotonic	Bernstein [2] Widder [39]
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	cyclically monotonic	Bernstein [4] Schoenberg [23]
$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	completely convex	Widder [38]
$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$ p columns	W_p-convex	Chapter I
$\begin{pmatrix} 1 & 0 & & 0 & 0 \\ 0 & 1 & \dots & 1 & 0 \\ 1 & 0 & & 0 & 0 \end{pmatrix}$ p columns (p even)	W_p^*-convex	Chapter II

CHAPTER I

A GENERALIZATION OF THE CLASS OF COMPLETELY CONVEX FUNCTIONS

1. Introduction.

In 1932, Whittaker [35] proved that an entire function $f(z)$ of exponential type $< \pi$ has a convergent Lidstone series expansion which is uniformly convergent in any finite region of the complex plane. Widder [38] showed that a necessary and sufficient condition for a function to have an absolutely convergent Lidstone series expansion is that it is the difference of two minimal completely convex functions. Later, in an attempt to synthesize the results of Bernstein on completely monotonic functions and the results of Widder on completely convex functions, several deep studies were made in different directions. We mention, in particular, the results of Boas and Polya [8] who showed, roughly speaking, that if $\{n_k\}\uparrow$ and $\{q_k\}$ are two sequences of nonnegative integers, and if $f^{(n_k)}(x)f^{(n_k+2q_k)}(x) \leq 0$ on an interval I, then f must necessarily coincide on this interval with an entire function of order one and finite type. In spite of the great generality of this result, a corresponding extension of Widder's interesting result was not attempted. Perhaps this could be attributed to the fact that the methods of Boas and Polya, as also of other authors who worked on this kind of problem, were very different from those of Widder.

Our object here is to introduce an extension of Lidstone series (called (p,L) series) and to obtain an analogue of Whittaker's result and then to obtain a necessary and sufficient condition that a function has an absolutely convergent (p,L) series expansion. In order to do so we give some preliminaries in §2 and introduce the class of W_p-convex functions and give a statement of the principal theorems. For $p = 2$, the W_p-convex functions become the class of completely convex functions of Widder.

In §3, we give a proof of Theorem I and in §4 we obtain some properties of the fundamental polynomials of the (p,L) series (see Definition 1). §5 deals with a boundary value problem which is useful in obtaining some estimates on the fundamental polynomials in §6. In §7, we use the results of §6 to obtain estimates for functions which are W_p-convex and complete the proof of Theorem II. In §8 we obtain some additional results on W_p-convex functions and prove Theorem III. The results of §8 and §9 together with the properties of minimal W_p-convex functions (see Definition 3), introduced in §10, enable us to give necessary and sufficient conditions for representation of a function by an absolutely convergent (p,L) series (Theorem IV) in §11.

2. Preliminaries and Statements of Main Results.

We define the sine functions of order p by

$$(2.1) \qquad M_{p,j}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!} \qquad (j=0,1,\ldots,p-1) .$$

Thus $M_{1,0}(t) = e^{-t}$, $M_{2,1}(t) = \sin t$,

$M_{3,0}(t) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{\frac{t}{2}}\cos\frac{t\sqrt{3}}{2}$. Then it is easy to see that

$$M^{(r)}_{p,j}(t) = \begin{cases} M_{p,j-r}(t) & , \quad (0 \le r \le j) \\ -M_{p,p+j-r}(t) & , \quad (j < r \le p) \end{cases}$$

and

$$(2.2) \qquad M_{p,j}(t) = \frac{\omega^{-(j/2)}}{p} \sum_{m=0}^{p-1} \omega^{-mj} e^{t\omega^{m+\frac{1}{2}}} \quad , \quad \omega = e^{2\pi i/p}$$

We shall require the addition formula (see [16]; p. 47)

$$(2.3) \quad M_{p,j}(x+y) = \sum_{k=0}^{j} M_{p,k}(x) M_{p,j-k}(y) - \sum_{k=j+1}^{p-1} M_{p,k}(x) M_{p,p+j-k}(y)$$

We denote the generalized hyperbolic functions of order p by

$$(2.4) \qquad N_{p,j}(t) = \sum_{n=0}^{\infty} \frac{t^{pn+j}}{(pn+j)!} \qquad (j=0,1,\ldots,p-1)$$

and observe that $N_{p,j}(t) = \omega^{j/2} M_{p,j}(t\omega^{-(1/2)})$.

Further, we denote the zeros $(\ne 0)$ of $M_{p,j}(t)$ by

$$(2.5) \qquad \lambda^{(j)}_1 < \lambda^{(j)}_2 < \ldots \qquad (j=0,1,\ldots,p-1)$$

and set $\lambda_k \equiv \lambda^{(p-1)}_k$; $\lambda^*_k \equiv \lambda_k$ $(k = 1,2,\ldots)$ when there is no chance of misunderstanding.

Mikusinski [16] has proved that <u>the zeros of</u> $M_{p,j}(t)$ <u>are simple and if</u> $0 \le j < k < p$, <u>the zeros of</u>

$M_{p,j}(t)$ *and* $M_{p,k}(t)$ *do not coincide. Further, if* $0 < \lambda_m^{(j)} < \lambda_{m+1}^{(j)}$ *are two consecutive zeros of* $M_{p,j}(t)$, *then there exists exactly one zero of* $M_{p,k}(t)$ *between* $\lambda_m^{(j)}$ *and* $\lambda_{m+1}^{(j)}$. We shall refer to this property as the *interlacing property* of the real zeros $(\neq 0)$ of the functions $M_{p,j}(t)$ $(j = 0,1,...,p-1)$.

We note that the moduli of the zeros of $N_{p,j}(t)$ are given by (2.5).

Lemma 2.1 (Mikusinski [16]). *Given an integer* $p \geq 2$, *the following properties hold:*

$$(2.6) \qquad \left[\frac{(p+j)!}{j!}\right]^{\frac{1}{p}} < \lambda_1^{(j)} < \left[\frac{2((p+j)!)}{j!}\right]^{\frac{1}{p}}$$

$$(2.7) \qquad \lambda_k^{(j)} < \lambda_k^{(p-1)} \qquad (j=0,1,...,p-2;k=1,2,...).$$

where $\lambda_k^{(j)}$ *is defined by* (2.5). *Furthermore,*

$$(2.8) \qquad \{\lambda_1^{(p-1)}\}_{p=2}^{\infty} \uparrow \quad ; \quad \lim_{p\to\infty} \lambda_1^{(p-1)} = \infty$$

$$(2.9) \qquad (-1)^k \, M_{p,j}(\lambda_k^{(p-1)}) > 0 \quad (j=0,1,...,p-2;k=1,2,...).$$

Proof. Inequalities (2.6) are due to Mikusinski. For $k = 1$, (2.7) follows from (2.6). By the interlacing property of the real zeros of the functions $M_{p,j}(t)$ $(j = 0,1,...,p-1)$ mentioned after formula (2.5), we have

(2.7) for all positive integers k. From the easily verified inequality $\lambda_1^{(p)} < \left[\frac{2((2p-1)!)}{(p-1)!}\right]^{\frac{1}{p}} < \left[\frac{(2p+1)!}{p!}\right]^{\frac{1}{p+1}} < \lambda_1^{(p+1)}$ it follows that the sequence (2.8) is strictly increasing and that $\lim_{p\to\infty} \lambda_1^{(p-1)} = \infty$. Now $M_{p,j}(x) \geq 0$ $(0 \leq x \leq \lambda_1^{(j)})$ and all the real zeros $(\neq 0)$ of $M_{p,j}(x)$ $(j = 0,1,\ldots,p-2)$ are simple and have the interlacing property with the zeros of $M_{p,p-1}(x)$. Therefore using (2.7) we have (2.9).

We now formulate

Theorem I. *Given an integer* $p \geq 2$, *the following representation holds for every entire function* $f(z)$ *of exponential type* $\tau < \lambda_1$:

$$(2.10)\quad f(z) = \sum_{n=0}^{\infty} \{f^{(pn)}(1)C_{pn}(z) + \sum_{\nu=0}^{p-2} f^{(pn+\nu)}(0)A_{pn+\nu}(z)\}$$

where $\lambda_1 \equiv \lambda_1^{(p-1)}$ *is defined by* (2.5) *and* $\{C_{pn}(z)\}_{n=0}^{\infty}$ *and* $\{A_{pn+j}(z)\}_{n=0}^{\infty}$, $(j = 0,1,\ldots,p-2)$ *are polynomials defined by the generating functions*:

$$(2.11)\qquad \sum_{n=0}^{\infty} t^{pn}\, C_{pn}(z) = \frac{N_{p,p-1}(zt)}{N_{p,p-1}(t)} \equiv \Psi_{p-1}(z,t^p)$$

$$(2.12)\qquad \sum_{n=0}^{\infty} t^{pn+j}\, A_{pn+j}(z) = N_{p,j}(zt) - N_{p,j}(t)\,\frac{N_{p,p-1}(zt)}{N_{p,p-1}(t)}$$

$$\equiv t^j\, \Psi_j(z,t^p) \qquad (j = 0,1,\ldots,p-2).$$

The series on the right in (2.10) converges to $f(z)$ for all z and the convergence is uniform on all bounded subsets of the plane.

Definition 1. We shall say that the series (2.10) is the p-Lidstone series (or (p,L) series) of f and that $\{C_{pn}(z)\}_{n=0}^{\infty}$ and $\{A_{pn+j}(z)\}_{n=0}^{\infty}$ $(j = 0,1,\ldots,p-2)$ are the fundamental polynomials of the (p,L) series.

Remark. The function $M_{p,p-1}(z\lambda_1)$ is of exponential type λ_1 and all the derivatives occurring in its (p,L) series vanish so that its (p,L) series is identically zero. Thus Theorem I yields a best possible result in the sense that λ_1 cannot be replaced by a larger number.

Definition 2. A real valued function f defined on $[a,b]$, is said to be W_p-convex if

(i) $f \in C^{\infty}[a,b]$,

(ii) $(-1)^k f^{(pk)}(x) \geq 0$, $(a \leq x \leq b;\ k=0,1,\ldots)$,

(iii) $(-1)^k f^{(pk+j)}(a) \geq 0$, $(j=1,2,\ldots,p-2;\ k=0,1,\ldots)$.

We now formulate

Theorem II. If f is W_p-convex on $0 \leq x \leq 1$ then f coincides on $[0,1]$ with a real entire function of exponential type not exceeding λ_1 (defined by (2.5)) and the (p,L) series representation holds if the function is of exponential type $< \lambda_1$.

We now give the following useful variation of Theorem II:

Theorem III. If $f(x)$ is W_p-convex in $a \le x \le b$ where $b-a > 1$ then $f(x)$ is an entire function of exponential type less than λ_1 and (2.10) holds for any z in the complex plane.

When p is an even integer, the first part of Theorem II is a special case of results of Boas and Polya (see [8]; Theorem 1, p. 407) except that we give a precise upper bound on the type of the entire function. When p is an odd integer however (say $p = 2m+1$), then our results do not follow as a special case of the results of Boas and Polya, because for $n_k = (2m+1)k$ there exists no sequence $\{q_k\}$ which will satisfy the hypothesis of Theorem 1 of Boas and Polya (p. 10). Furthermore, the conditions (iii) of Definition 2 are imposed at the end-point of the interval in question, whereas a result of Boas and Polya ([8], p. 423) imposes conditions on the function on subintervals about the midpoint of the interval. Our method of proof is close to that of Widder ([37], [38]).

Theorem II gives a sufficient condition for a function to have a (p,L) series representation but it is not necessary as seen from the example of the function $N_{p,p-1}(x)$ which is not W_p-convex and yet has the (p,L) series representation $N_{p,p-1}(x) = N_{p,p-1}(1) \sum_{n=0}^{\infty} C_{pn}(x)$.

Also, $M_{p,p-1}(x\lambda_1)$ is W_p-convex on $[0,1]$, yet it has no (p,L) series representation. In order to obtain a necessary and sufficient condition, we follow Widder and introduce the class of minimal W_p-convex functions.

Definition 3. *A real valued function $f(x)$ defined on $0 \le x \le 1$ is minimal W_p-convex on $[0,1]$ if it is W_p-convex on $[0,1]$ and if $f(x) - \varepsilon M_{p,p-1}(x\lambda_1)$ is not W_p-convex on $[0,1]$ for any positive ε.*

This leads us to formulate

Theorem IV. *A necessary and sufficient condition that $f(x)$ be represented by an absolutely convergent (p,L) series is that it is the difference of two minimal W_p-convex functions on $0 \le x \le 1$.*

3. Proof of Theorem I.

Setting $f(z) = e^{zt}$ in (2.10) we get the formal (p,L) series representation of e^{zt}, so that

$$e^{zt} = \sum_{j=0}^{p-2} t^j \Psi_j(z,t^p) + e^t \Psi_{p-1}(z,t^p) \quad . \tag{3.1}$$

Replacing t successively in this relation by ωt, $\omega^2 t$, ..., $\omega^{p-1}t$, with $\omega = e^{2\pi i/p}$, and observing that $\Psi_j(z,t^p)$ remains unchanged, we get the following system

of equations in Ψ_j :

$$(3.2) \qquad e^{\omega^m zt} = \sum^{p-2} (\omega^m t)^j \Psi_j + e^{\omega^m t}\Psi_{p-1} \qquad (m=0,1,\ldots,p-1) .$$

In order to obtain Ψ_j , we multiply the $(m+1)^{th}$ equation in (3.2) by ω^{-mj} $(m = 0,1,\ldots,p-1)$ and add. Now, keeping in mind the easily verified identities

$$(3.3) \qquad \sum_{m=0}^{p-1} \omega^{(\nu-j)m} = \begin{cases} 0, & \nu \neq j \pmod p \\ p, & \nu = j \pmod p \end{cases}$$

$$(3.4) \qquad \sum_{m=0}^{p-1} \omega^{-mj} e^{\omega^m t} = pN_{p,j}(t) \qquad (j=0,1,\ldots,p-1) ,$$

we obtain (2.11) and (2.12) .

The polya representation of an entire function $f(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$ of finite type is given by

$$(3.5) \qquad f(z) = \frac{1}{2\pi i} \int_\Gamma e^{zt} F(t)dt$$

where $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}$ is the Borel Transform of $f(z)$ and Γ is a contour surrounding the conjugate indicator diagram $D(f)$ of f , i.e. the convex hull of the set of singularities of $F(t)$. From (2.11) and (2.12) we have that the right hand side of (3.1) is regular in all circles $|t| = \rho$ where $\rho < \lambda_1$. Therefore, the series given by $\Psi_0, \Psi_1, \ldots, \Psi_{p-1}$ converge uniformly in any compact subset of the disk $|t| < \lambda_1$.

Now if f is of exponential type τ, it is well known (see e.g. [7], p. 8) that $D(f)$ lies inside the disk $|t| \le \tau$. Therefore Γ can be taken to be any circle $|t| = \rho$ where $\tau < \rho < \lambda_1$. The proof of Theorem I is completed by applying the kernel expansion method (see [7], p. 10) with e^{zt} as kernel.

4. <u>Properties of Fundamental Polynomials of</u> (p,L) <u>Series</u>.

Consider the linear operator L defined on $C^p[0,1]$ by

$$L(f) = f(x) - \left[f(1)C_0(x) + \sum_{j=0}^{p-2} f^{(j)}(0)A_j(x)\right] \tag{4.1}$$

where $C_0(x)$ and $A_j(x)$ $(j = 0,1,\ldots,p-2)$ are the polynomials occurring in (2.10). Since $L(P) = 0$ for any polynomial $P(x)$ of degree $\le p-1$, we have by Peano's theorem [12]

$$L(f) = \int_0^1 K_1(x,t)\, f^{(p)}(t)dt \tag{4.2}$$

where

$$(p-1)!\; K_1(x,t) = L_x[(x-t)_+^{p-1}] \tag{4.3}$$

with

$$(x-t)_+^{p-1} = \begin{cases} 0 & , \quad t > x \\ (x-t)^{p-1} & , \quad t \le x . \end{cases}$$

Setting $f(x) = 1, x, x^2, \ldots, x^{p-1}$ successively in (4.1), we easily have

$$(4.4) \qquad \begin{cases} j!A_j(x) = x^j - x^{p-1} & (j=0,1,\ldots,p-2) \\ C_0(x) = x^{p-1} & \end{cases}$$

From (4.1) and (4.3) we have

$$(4.5) \qquad (p-1)!\ K_1(x,t) = \begin{cases} (x-t)^{p-1} - (1-t)^{p-1}x^{p-1} & (0 \le t < x \le 1) \\ -(1-t)^{p-1}x^{p-1} & (0 \le x \le t \le 1) \end{cases}$$

Also $K_1(x,t) = K_1(1-t,1-x)$. Now $K_1(x,t)$ is seen to be the Green's function for the differential system

$$(4.6) \qquad \begin{cases} y^{(p)}(x) = \phi(x) \\ y(1) = 0 \ ;\ y(0) = y'(0) = \ldots = y^{(p-2)}(0) = 0 \ . \end{cases}$$

where $\phi(x)$ is any function continuous on $0 \le x \le 1$, so that

$$(4.7) \qquad y(x) = \int_0^1 K_1(x,t)\phi(t)dt$$

is the unique solution of the system (4.6). Since $A_{pn+j}(x)$ satisfies (4.6) with $\phi(x) = A_{p(n-1)+j}(x)$ we have

$$(4.8) \qquad A_{pn+j}(x) = \int_0^1 K_1(x,t)A_{p(n-1)+j}(t)dt$$

$$= \int_0^1 K_n(x,t)A_j(t)dt \qquad (j=0,1,\ldots,p-2;\ n=1,2,\ldots)$$

where we set

$$(4.9) \qquad K_n(x,t) = \int_0^1 K_1(x,u)\ K_{n-1}(u,t)du \qquad (n=2,3,\ldots)\ .$$

Similarly

$$C_{pn}(x) = \int_0^1 K_n(x,t)C_0(t)dt \qquad (n=1,2,\ldots) . \tag{4.10}$$

Thus we have

<u>Lemma 4.1</u>. <u>If</u> $f(x)$ <u>belongs to</u> $C^{pn}[0,1]$ <u>then</u>

$$f(x) = \sum_{k=0}^{n-1} f^{(pk)}(1)C_{pk}(x) + \tag{4.11}$$

$$+ \sum_{j=0}^{p-2} \sum_{k=0}^{n-1} f^{(pk+j)}(0)A_{pk+j}(x) + R_n(x,f)$$

<u>where</u>

$$R_n(x,f) = \int_0^1 K_n(x,t)f^{(pn)}(t)dt \tag{4.12}$$

<u>with</u> $K_n(x,t)$ <u>given by</u> (4.5) <u>and</u> (4.9).

<u>Proof</u>. For $n = 1$, (4.11) is given by (4.1) and (4.2). The proof is completed by induction on n .

<u>Lemma 4.2</u>. <u>The following inequalities hold for</u> $0 \le x \le 1$

$$(-1)^n K_n(x,t) \ge 0 \qquad (0\le t\le 1;\ n=1,2,\ldots) , \tag{4.13}$$

$$\begin{cases} (-1)^n C_{pn}(x) \ge 0 & (n=0,1,2,\ldots) , \\ (-1)^n A_{pn+j}(x) \ge 0 & (j=0,1,\ldots,p-2) . \end{cases} \tag{4.14}$$

<u>Proof</u>. For $n = 1$, and $x \le t$ (4.13) is clear from (4.5). For $x > t$, consider the expression

$(x-t)^{p-1} - (1-t)^{p-1}x^{p-1} = (1-t)^{p-1}F(t)$, where

$F(t) = \left(\frac{x-t}{1-t}\right)^{p-1} - x^{p-1}$, x being fixed. Then

$F'(t) = \frac{(p-1)(x-t)^{p-2}(x-1)}{(1-t)^p} \le 0$, so that $F(t)$ is monotone decreasing for $0 \le t < x \le 1$ and hence assumes its maximum at $t = 0$. Since $F(0) = 0$, $F(t) \le 0$, $(0 \le t \le x)$; and hence $K_1(x,t) \le 0$, $(0 \le t \le x \le 1)$. For $n > 1$, (4.13) is proved from (4.9) . Also, (4.14) is immediate from (4.8), (4.10) and (4.13).

We supplement (4.14) with

<u>Lemma 4.3</u>. <u>The fundamental polynomials</u> $C_{pn}(x)$, $A_{pn+j}(x)$ $(j = 0,1,\ldots,p-2)$ <u>have no zeros in the interval</u> $0 < x < 1$.

<u>Proof</u>. We shall prove the Lemma for the polynomials $A_{pn}(x)$. The proof for the other fundamental polynomials is identical. From (4.14) we have $(-1)^n A_{pn}(x) \ge 0$ $(n = 0,1,2,\ldots)$. Also, $A_{pn}(0) = A'_{pn}(0) = \ldots = A_{pn}^{(p-2)}(0) = 0$, $(n = 1,2,\ldots)$. Since $A_0 = 1 - x^{p-1}$, the Lemma is true for $n = 0$. Suppose it is true for $n = k-1$. $A_{pn}(x)$ $(n = 1,2,\ldots)$ has a simple zero at $x = 1$ and a zero of order $p-1$ at $x = 0$. Assume that x_o $(0 < x_o < 1)$ is a zero of $A_{pk}(x)$. Then x_o must be a zero of even order (at least two). Therefore, under

our assumption, $A_{pk}(x)$ has (at least) $p+2$ zeros in $0 \le x \le 1$. Applying Rolle's Theorem p times, $A_{pk}^{(p)}(x) = A_{p(k-1)}(x)$ has (at least) two zeros in the interval $0 < x < 1$, contradicting the inductive assumption.

5. A Boundary Value Problem.

Consider the boundary value problem

$$(5.1) \qquad \begin{cases} y^{(p)} + \lambda^p y = 0 \\ y(0) = y'(0) = \ldots = y^{(p-2)}(0) = y(1) = 0 \end{cases}$$

and the adjoint problem

$$(5.2) \qquad \begin{cases} (-1)^p z^{(p)} + \lambda^p z = 0 \\ z(1) = z'(1) = \ldots = z^{(p-2)}(1) = z(0) = 0 . \end{cases}$$

Then the real eigenvalues $\lambda_1 < \lambda_2 < \ldots$ are the zeros of $M_{p,p-1}(x)$ (defined in (2.5)). The eigenfunctions of (5.1) are $\{M_{p,p-1}(x\lambda_k)\}_{k=1}^{\infty}$ and those of (5.2) are $\{M_{p,p-1}(\lambda_k - x\lambda_k)\}_{k=1}^{\infty}$.

There is considerable literature on the problem of expansion of a function in terms of the eigenfunctions of the above boundary value problem, which is classified as non-regular and separable [17].

Lemma 5.1. The following biorthogonal property holds:

$$(5.3)\quad \int_0^1 M_{p,p-1}(x\lambda_k)\, M_{p,p-1}(\lambda_j - x\lambda_j)\,dx = \begin{cases} 0 & , \; j \neq k \\ \dfrac{-M_{p,p-2}(\lambda_k)}{p} & , \; j = k \end{cases}$$

where $\lambda_1 < \lambda_2 < \ldots$ are the real zeros of $M_{p,p-1}(z)$.

Proof. If $j \neq k$, formula (5.3) is easy to verify. To verify it for $j = k$ we set $y = M_{p,p-1}(x\lambda_k)$, $z = M_{p,p-1}(\lambda_k - x\lambda_k)$. Then from (5.1) and (5.2) we have

$$(5.4)\quad -2(\lambda_k)^p \int_0^1 yz\,dx = \int_0^1 [y^{(p)}z + (-1)^p z^{(p)}y]\,dx$$

$$= (-1)^j \int_0^1 [(-1)^p z^{(p-j)} y^{(j)} + y^{(p-j)} z^{(j)}]\,dx$$

$(j = 1,2,\ldots,p-1)$ so that

$$-2(p-1)(\lambda_k)^p \int_0^1 yz\,dx$$

$$= \int_0^1 \sum_{j=0}^{p-1} (-1)^j [(-1)^p z^{(p-j)} y^{(j)} + y^{(p-j)} z^{(j)}]\,dx$$

$$= 2(\lambda_k)^p \int_0^1 \sum_{j=1}^{p-1} M_{p,j-1}(\lambda_k - x\lambda_k)\, M_{p,p-j-1}(x\lambda_k)\,dx \quad .$$

Using the addition formula (2.3) for $j = p-2$ we obtain (5.3). This completes the proof of (5.3) for $j = k$.

<u>Remark 1</u>. Formula (5.3) may be generalized as follows. If $\lambda_k^{(j)}$ $(j = 0,1,\ldots,p-1;\ k = 1,2,\ldots)$ denote the real zeros $(\neq 0)$ of $M_{p,j}(x)$, then

$$\int_0^1 M_{p,j}(x\lambda_k^{(j)})\, M_{p,p-1}\left(\lambda_\ell^{(j)}-x\lambda_\ell^{(j)}\right) dx = \begin{cases} 0 & , \ \ell \neq k \\ \dfrac{-M'_{p,j}(\lambda_k^{(j)})}{p} & , \ \ell = k \end{cases}$$

<u>Remark 2</u>. By Lemma 5.1, a formal expansion of a function $f(x)$ can be written down

$$(5.5) \qquad f(x) = \sum_{k=1}^{\infty} a_k M_{p,p-1}(x\lambda_k)$$

where $a_k = \int_0^1 f(x) M_{p,p-1}(\lambda_k - x\lambda_k)\, dx \Big/ \left(-\dfrac{M_{p,p-2}(\lambda_k)}{p}\right)$.

However, regarding the convergence problem, we know from a result of Ward [33] that the right side of (5.5) converges uniformly to $f(x)$ for $0 \leq x \leq x_o$ for any $x_o < 1$ if $f(x)$ is of the form $f(x) = x^{p-1}\psi(x^p)$ where $\psi(x^p)$ is a convergent power series in x^p . Since $C_0(x) = x^{p-1}$ is of this form, and since it is easily verified that

$$(5.6) \qquad \int_0^1 x^{p-1} M_{p,p-1}(\lambda_k - x\lambda_k)\, dx = \frac{1}{\lambda_k}$$

we have the convergent expansion

$$(5.7) \qquad C_0(x) = -p \sum_{k=1}^{\infty} \frac{M_{p,p-1}(x\lambda_k)}{\lambda_k M_{p,p-2}(\lambda_k)} \qquad (0 \leq x \leq x_o) \ .$$

Obviously, since $C_0(1) = 1$ and since $M_{p,p-1}(\lambda_k) = 0$ $(k = 1,2,\ldots)$, the right hand side of (5.7) cannot converge to $C_0(x)$ at $x = 1$. Now $C_{pn}(x)$ satisfies the differential system (4.6) with $\phi(x) = C_{p(n-1)}(x)$. Thus, we have

$$(5.8) \qquad C_{pn}(x) = (-1)^{n-1} p \sum_{k=1}^{\infty} \frac{M_{p,p-1}(x\lambda_k)}{M_{p,p-2}(\lambda_k)(\lambda_k)^{pn+1}}$$

Since $C_{pn}(1) = 0$ $(n = 1,2,\ldots)$, by a theorem of Ward ([33], Theorem 4) the right side of (5.8) converges uniformly to $C_{pn}(x)$ in $0 \le x \le 1$.

6. Estimates on the Fundamental Polynomials.

It is our object to show here that for large n the first term in the formal expansion of the fundamental polynomials $\{C_{pn}(x)\}$ and $\{A_{pn+j}(x)\}$ $(j = 0,1,\ldots,p-2)$ in terms of the eigenfunctions $M_{p,p-1}(x\lambda_k)$ serves as a good approximation to these polynomials for $0 \le x \le 1$. However, the formal eigenfunction expansion of these polynomials may not necessarily converge to the polynomial on $0 \le x \le 1$ (see Ward [33]).

In the remaining sections of Chapter I, we use B to denote suitable constants (not necessarily the same), which are independent of n and x, $(0 \le x \le 1)$, unless otherwise stated.

<u>Lemma 6.1</u>. <u>For</u> $0 \le x \le 1$; $n = 0,1,\ldots$ <u>we</u> <u>have</u>

$$(6.1) \qquad \left| (-1)^{n+1} C_{pn}(x) - \frac{pM_{p,p-1}(x\lambda_1)}{M_{p,p-2}(\lambda_1)(\lambda_1)^{pn+1}} \right| < \frac{B}{(\lambda_2)^{pn+1}}$$

$$(6.2) \qquad \left| (-1)^{n} A_{pn+j}(x) - \frac{pM_{p,j}(\lambda_1)\, M_{p,p-1}(x\lambda_1)}{M_{p,p-2}(\lambda_1)(\lambda_1)^{pn+j+1}} \right| < \frac{B}{\left(\frac{\lambda_1+\lambda_2}{2}\right)^{pn+1}}$$

$$(j = 0,1,\ldots,p-2) \; .$$

<u>Proof</u>. From (5.8) we have

$$\left| (-1)^{n+1} C_{pn}(x) - \frac{pM_{p,p-1}(x\lambda_1)}{M_{p,p-2}(\lambda_1)(\lambda_1)^{pn+1}} \right|$$

$$= \frac{1}{(\lambda_2)^{pn+1}} \left| \sum_{k=2}^{\infty} \left(\frac{\lambda_2}{\lambda_k}\right)^{pn+1} \frac{M_{p,p-1}(x\lambda_k)}{M_{p,p-2}(\lambda_k)} \right|$$

$$\le \frac{B_o}{(\lambda_2)^{pn+1}} \sum_{k=2}^{\infty} \left(\frac{\lambda_2}{\lambda_k}\right)^{pn+1} \le \frac{B}{(\lambda_2)^{pn+1}}$$

by (2.5) and since it is easily shown from (2.2) that $\left|\frac{M_{p,p-1}(x\lambda_k)}{M_{p,p-2}(\lambda_k)}\right| \le B_o$ $(0 \le x \le 1 \; ; \; k = 2,3,\ldots)$. This proves inequality (6.1).

Consider the circles $\Gamma_o : |t| = \frac{\lambda_1}{2} = r_o$ and $\Gamma_1 : |t| = \frac{1}{2}(\lambda_1+\lambda_2) = r_1$ where λ_1 and λ_2 are defined by (2.3).

Define

$$(6.3)\qquad A_{pn+j,k}(x) = \frac{1}{2\pi i}\int_{\Gamma_k} t^{-pn-1}\Psi_j(x,t^p)\,dt$$

$(0 \le x \le 1\ ;\ \ k = 0,1)$, where $\Psi_j(x,t^p)$ $(j = 0,1,\ldots,p-2)$ is defined by (2.12). Thus we have $A_{pn+j,0}(x) = A_{pn+j}(x)$ and since $N_{p,p-1}(t)$ is uniformly bounded away from zero when $t \in \Gamma_1$, we have

$$(6.4)\qquad \left|A_{pn+j,1}(x)\right| =$$

$$= \left|\frac{1}{2\pi i}\int_{\Gamma_1} t^{-pn-j-1}\left[N_{p,j}(x,t) - \frac{N_{p,j}(t)\ N_{p,p-1}(xt)}{N_{p,p-1}(t)}\right]dt\right|$$

$$\le \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{(1+|x|)r_1}\,d\theta}{(r_1)^{pn+j+1}\left|N_{p,p-1}(r_1e^{i\theta})\right|} \le \frac{B}{(r_1)^{pn+j}} \qquad (0\le x\le 1)\ .$$

It is easily verified that the residue of $t^{-pn-1}\Psi_j(x,t^p)$ at $t = \omega^{\nu+\frac{1}{2}}\lambda_1$ $(\nu = 0,1,\ldots,p-1)$ is

$$(6.5)\qquad R_{n,j}(x) = \frac{(-1)^{n+1}M_{p,j}(\lambda_1)\ M_{p,p-1}(x\lambda_1)}{M_{p,p-2}(\lambda_1)(\lambda_1)^{pn+j+1}}\ .$$

Therefore, we have

$$(6.6)\qquad A_{pn+j,1}(x) - A_{pn+j}(x) = pR_{n,j}(x)\ .$$

Using (6.5) and (6.6) we have

$$(6.7)\qquad \left|(-1)^n A_{pn+j}(x) - \frac{pM_{p,j}(\lambda_1)\ M_{p,p-1}(x\lambda_1)}{M_{p,p-2}(\lambda_1)(\lambda_1)^{pn+j+1}}\right| = \left|A_{pn+j,1}(x)\right|\ .$$

Then (6.2) follows from (6.7) and (6.4).

Remark. It should be noted that if the same techniques that are used to prove inequality (6.2) are applied to the polynomials $C_{pn}(x)$, then the inequality

$$\left| (-1)^{n-1} C_{pn}(x) - \frac{pM_{p,p-1}(x\lambda_1)}{M_{p,p-2}(\lambda_1)(\lambda_1)^{pn+1}} \right| < \frac{B}{(r_1)^{pn+1}}$$

is obtained. However, using (5.8), the better estimate (6.1) is obtained.

Lemma 6.2. *There exist constants* B *such that for* $0 \le x \le 1$; $n = 1,2,\ldots$

(6.8) $$0 \le (-1)^n C_{pn}(x) \le \frac{B}{(\lambda_1)^{pn}}$$

(6.9) $$0 \le (-1)^n A_{pn+j}(x) \le \frac{B}{(\lambda_1)^{pn}} \qquad (j=0,1,\ldots,p-2).$$

Proof. From (6.1) we have

$$0 \le (-1)^n C_{pn}(x) \le \frac{B}{(\lambda_2)^{pn+1}} + \left| \frac{pM_{p,p-1}(x\lambda_1)}{M_{p,p-2}(\lambda_1)(\lambda_1)^{pn+1}} \right| \le \frac{B}{(\lambda_1)^{pn}}$$

Since $\lambda_1 < \lambda_2$, and $M_{p,p-1}(x\lambda_1)$ is uniformly bounded for $0 \le x \le 1$. We get (6.9) similarly from (6.2).

<u>Lemma 6.3</u> <u>For any fixed</u> x_o <u>such that</u> $0 < x_o < 1$ <u>there exist constants</u> B <u>such that</u>

$$(6.10) \qquad (-1)^n C_{pn}(x_o) \geq \frac{B}{(\lambda_1)^{pn}} \qquad (n=1,2,\ldots)$$

$$(6.11) \qquad (-1)^n A_{pn+j}(x_o) \geq \frac{B}{(\lambda_1)^{pn}} \quad (j=0,1,\ldots,p-2; n=1,2,\ldots) .$$

<u>Proof</u>. We shall prove (6.10) . From (6.1) we have

$$(6.12) \qquad \lim_{n\to\infty} \frac{(-1)^{n-1} C_{pn}(x_o)\, M_{p,p-2}(\lambda_1)(\lambda_1)^{pn}}{M_{p,p-1}(x_o\lambda_1)} = \frac{p}{\lambda_1} ,$$

from which (6.10) follows easily. The proof of (6.11) follows in an analogous way.

<u>Lemma 6.4</u> <u>For</u> $0 \leq x \leq 1$, $n = 1,2,\ldots$ <u>we have</u>

$$(6.13) \qquad 0 \leq (-1)^n \int_0^1 K_n(x,t)\,dt \leq \frac{B}{(\lambda_1)^{pn}}$$

<u>where</u> $K_n(x,t)$ <u>is defined by</u> (4.9) <u>and</u> (4.5) .

<u>Proof</u>. Since $A_0(x) + C_0(x) = 1$ we have

$0 \leq (-1)^n \int_0^1 K_n(x,t)\,dt = (-1)^n [A_{pn}(x) + C_{pn}(x)]$ and (6.13) follows at once from Lemma 6.2.

7. Estimates for W_p-convex Functions. Proof of Theorem II.

Lemma 7.1 (Hadamard). *If $g(x)$ belongs to $C^{(p)}(I)$ where I is a closed interval of length α and if*

(7.1) $$|g(x)| \le M_o \ ; \quad |g^{(p)}(x)| \le M_p \ , \quad x \in I \ ,$$

then throughout the interval I

(7.2) $$|g^{(j)}(x)| \le \left(\frac{ep^2}{j}\right)^j [\alpha^{-j} M_o + \frac{\alpha^{p-j}}{p!} M_p] \qquad (1 \le j \le p-1) \ .$$

For a proof of this Lemma see [10], p. 13. We shall now prove

Lemma 7.2. *If f is W_p-convex on $0 \le x \le 1$ then for sufficiently large k we have*

(7.3) $$(-1)^k f^{(pk)}(1) \le B(\lambda_1)^{pk}$$

(7.4) $$(-1)^k f^{(pk+j)}(0) \le B(\lambda_1)^{pk} \qquad (j=0,1,\ldots,p-2)$$

where $\lambda_1 \equiv \lambda_1^{(p-1)}$ is defined by (2.5).

Proof. From the definition of W_p-convex functions and Lemma 4.2 every term on the right hand side in (4.11) is nonnegative so that

$$0 \le f^{(pk)}(1) C_{pk}(x) \le f(x)$$

$$0 \le f^{(pk+j)}(0) A_{pk+j}(x) \le f(x) \qquad (j=0,1,\ldots,p-2) \ .$$

If we choose $x = \frac{1}{2}$ and apply Lemma 6.3 to the above inequalities we have (7.3) and (7.4).

<u>Lemma 7.3</u>. <u>If</u> (i) $f^{(j)}(0) \ge 0$ $(j = 1,2,\dots,p-2)$, <u>and</u> (ii) $f(x) \ge 0$, $-f^{(p)}(x) \ge 0$ $(0 \le x \le 1)$ <u>then</u>

$$f(x) \ge f(x_o)x^{p-1} \qquad (0\le x\le x_o) \ , \tag{7.5}$$

$$f(x) \ge f(x_o)(1-x^{p-1}) \qquad (x_o\le x\le 1) \ . \tag{7.6}$$

<u>Proof</u>. Setting $n = 1$ in (4.11) and replacing the node 1 by x_o $(0 < x_o \le 1)$ yields

$$f(x) = \sum_{j=0}^{p-2} f^{(j)}(0)A_j\left(\frac{x}{x_o}\right) + f(x_o)C_0\left(\frac{x}{x_o}\right) + R(f,x,x_o) \tag{7.7}$$

where $C_0(x)$ and $A_j(x)$ $(j = 0,1,\dots,p-2)$ are defined by (4.4) and, by Peano's theorem [12]

$$R(f,x,x_o) = \int_0^{x_o} K_1(x,x_o,t)f^{(p)}(t)dt \quad \text{with}$$

$$(p-1)!K_1(x,x_o,t) = (x-t)_+^{p-1} - (x_o-t)^{p-1}\left(\frac{x}{x_o}\right)^{p-1} \quad (0\le x\le x_o; 0\le t\le x_o).$$

Using (ii) of the hypothesis, it is easily seen that all the terms on the right side of (7.7) are nonnegative, and we have (7.5).

To obtain (7.6) we define

$$L^*(f) \equiv f(x) - \left[f(x_o)D_0(x) + f(1)D_1(x) + \sum_{j=1}^{p-2} f^{(j)}(0)E_j(x)\right] = R^*(f,x,x_o) \tag{7.8}$$

where for $x_o \le x \le 1$, we have

$$(7.9) \quad \begin{cases} D_0(x) = \dfrac{1-x^{p-1}}{1-x_o^{p-1}} \geq 0 \; ; \quad D_1(x) = \dfrac{x^{p-1}-x_o^{p-1}}{1-x_o^{p-1}} \geq 0 \\ j!E_j(x) = x^j - 1 + \left(\dfrac{1-x_o^j}{1-x_o^{p-1}}\right)(1-x^{p-1}) \geq 0 \quad . \end{cases}$$

It is easily verified that $L^*(P) = 0$ for all polynomials $P(x)$ of degree $\leq n-1$. Again, using Peano's theorem [12] we have

$$(7.10) \qquad R^*(f,x,x_o) = \int_0^1 K^*(x,x_o,t)f^{(p)}(t)dt$$

with $(p-1)!K^*(x,x_o,t) = L_x^*[(x-t)_+^{p-1}] \leq 0$ for $x_o \leq x \leq 1$, $0 \leq t \leq 1$, (see p. 41 **) and with L^* defined by (7.8). Using (ii) of the hypothesis it can be easily seen that $R^*(f,x,x_o) \geq 0$. So from (7.8) and (7.9) we have

$$(7.11) \qquad f(x) = f(x_o)D_0(x) + f(1)D_1(x) + \\ + \sum_{j=1}^{p-2} f^{(j)}(0)E_j(x) + R^*(f,x,x_o) \quad ,$$

where every term on the right side of (7.11) is nonnegative. Therefore, inequality (7.6) is established and this proves the lemma.

<u>Lemma 7.4</u>. <u>If</u> (i) $f^{(j)}(0) \geq 0$ $(j = 1,2,\ldots,p-2)$, <u>and</u> (ii) $f(x) \geq 0$, $-f^{(p)}(x) \geq 0$ $(0 \leq x \leq 1)$, <u>then for</u> $0 \leq a < b \leq \left(\frac{1}{2}\right)^{\frac{1}{p-1}}$ <u>we have</u>

$$(7.12) \qquad f(x) \leq \frac{p}{b^p-a^p}\int_a^b f(x)dx \qquad (a \leq x \leq b) \; .$$

Proof. Let $f(x_o) = \max_{a \le x \le b} f(x)$. Then from (7.5) and (7.6) we have

$$\int_a^b f(x)dx \ge f(x_o)\left[\int_a^{x_o} x^{p-1}dx + \int_{x_o}^b (1-x^{p-1})dx\right]$$

$$= f(x_o)\left[\left(b-\frac{2b^p}{p}\right) + \frac{b^p-a^p}{p} - \left(x_o-\frac{2x_o^p}{p}\right)\right] \ge f(x_o)\left(\frac{b^p-a^p}{p}\right) ,$$

since $x - \frac{2x^p}{p}$ is increasing for $0 \le x \le \left(\frac{1}{2}\right)^{\frac{1}{p-1}}$.

8. Proof of Theorem II.

Using the properties $M_{p,p-1}(x\lambda_1)$ and using integration by parts we obtain

$$\int_0^1 f(x)M_{p,p-1}(\lambda_1-x\lambda_1)dx = \frac{f(1)}{\lambda_1} - \sum_{j=0}^{p-2} \frac{f^{(j)}(0)}{(\lambda_1)^{j+1}} M_{p,j}(\lambda_1) -$$

$$- \left(\frac{1}{\lambda_1}\right)^p \int_0^1 f^{(p)}(x)M_{p,p-1}(\lambda_1-x\lambda_1)dx$$

where λ_1 is defined by (2.5). Since $f(x)$ is W_p-convex $f(1) \ge 0$, $f^{(j)}(0) \ge 0$ and, by Lemma 2.1, $M_{p,j}(\lambda_1) < 0$ $(j = 0,\ldots,p-2)$, we have

$$(8.1) \quad \int_0^1 f(x)M_{p,p-1}(\lambda_1-x\lambda_1)dx \ge -\left(\frac{1}{\lambda_1}\right)^p \int_0^1 f^{(p)}(x)M_{p,p-1}(\lambda_1-x\lambda_1)dx .$$

From the definition of W_p-convex functions, it is obvious that $-f^{(p)}(x)$ is also W_p-convex so that on successively using the inequality (8.1) we have

$$(-1)^k\left(\frac{1}{\lambda_1}\right)^{pk}\int_0^1 f^{(pk)}(x)M_{p,p-1}(\lambda_1-x\lambda_1)dx$$

$$\leq \int_0^1 f(x)M_{p,p-1}(\lambda_1-x\lambda_1)dx \equiv A_p \quad .$$

If $0 < a < b < \left(\frac{1}{2}\right)^{\frac{1}{p-1}}$ then a fortiori

$$(-1)^k\left(\frac{1}{\lambda_1}\right)^{pk}\int_a^b f^{(pk)}(x)M_{p,p-1}(\lambda_1-x\lambda_1) \leq A_p \quad .$$

Elementary geometric considerations show that

$\min\limits_{a\leq x\leq b} M_{p,p-1}(\lambda_1-x\lambda_1) = D > 0$, so that

$$(-1)^k\int_a^b f^{(pk)}(x)dx < \frac{A_p(\lambda_1)^{pk}}{D} \quad .$$

Hence, by Lemma 7.4, we have

$$(8.2) \qquad (-1)^k f^{(pk)}(x) \leq \frac{pA_p(\lambda_1)^{pk}}{(b^p-a^p)D} \qquad (a\leq x\leq b) \quad .$$

From Lemma 7.1 we see that for $j = 0,1,...,p-1$ $f^{(pk+j)}(x) = O\left((\lambda_1)^{pk}\right)$ uniformly in $[a,b]$, as $k \to \infty$. Thus, we have $f^{(n)}(x) = O\left((\lambda_1)^n\right)$ uniformly in $[a,b]$, as $n \to \infty$, which shows that $f(x)$ is entire and of exponential type $\leq \lambda_1$, which completes the proof of the first part of Theorem II. The second part of Theorem II follows from Theorem I.

An interesting consequence of Theorem II is the following:

Corollary 8.1. *If* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *where the coefficients are real and such that* $(-1)^n a_{pn+\nu} > 0$ $(\nu = 0,1,\ldots,p-2;\ n = 0,1,\ldots)$ *and*

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} > 0 \quad , \tag{8.3}$$

then in any interval $0 \le x \le \delta$, *however small, there is some* n *such that one of the derivatives of order* $pn, pn+1, \ldots, pn+p-2$ *changes sign.*

Proof. Suppose no such n exists. Then f is W_p-convex on $0 \le x \le \delta$, hence entire, which contradicts (8.3).

Remark. If, in Theorem II, we consider the case $p = 3$, then for $f(x) = e^{-cx}$ $(c > 0)$ we have $(-1)^k f^{(3k)}(x) \ge 0$; $(-1)^k f^{(3k+1)}(a) < 0$, $(k = 0,1,\ldots)$, so that condition (iii) cannot be waived in the definition of W_p-convex functions.

** The inequality for $K^*(x,x_o,t)$ on page 38 follows from Theorem II of Birkhoff (Trans. A.M.S. 7 (1906) 107-136).

9. The (p,L) Series and W_p-Convex Functions. Proof of Theorem III.

We shall use the following theorem in §10 to obtain necessary and sufficient conditions for representation of a function by a (p,L) series.

Theorem 9.1. *If the series*

$$(9.1)\qquad c_0C_0(x) + a_0A_0(x) + \dots + a_{p-2}A_{p-2}(x) + c_pC_p(x) + \dots$$

converges for a single value x_o $(0 < x_o < 1)$ *then it converges uniformly in* $0 \le x \le 1$ *to a function* $f(x)$. *Furthermore, the series*

$$(9.2)\qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_1)^{pn}} \left[c_{pn} - \sum_{j=0}^{p-2} \frac{M_{p,j}(\lambda_1)}{(\lambda_1)^j} a_{pn+j} \right]$$

converges and we have

$$(9.3)\qquad f^{(pk)}(x) = c_{pk}C_0(x) + a_{pk}A_0(x) + \dots$$

$$+ a_{pk+p-2}A_{p-2}(x) + c_{(p+1)k}C_p(x) + \dots$$

for $0 \le x \le 1$.

Proof. With suitable modifications, the proof follows Widder's method for the case $p = 2$, (see [38]; Theorem 5.2, p. 392).

Now we give some results for W_p-convex functions.

<u>Lemma 9.1</u>. <u>If</u> $f(x)$ <u>is</u> W_p-<u>convex in</u> $0 \le x \le 1$ <u>then there is a constant</u> M <u>such that</u>

$$(9.4) \qquad \begin{cases} 0 \le (-1)^k f^{(pk)}(x) \le B\left(\dfrac{\lambda_1}{x}\right)^{pk} \\ 0 \le (-1)^k f^{(pk)}(x) \le B\left(\dfrac{\lambda_1}{1-x}\right)^{pk} \end{cases} \qquad (k \to \infty) .$$

If $f(x)$ is W_p-convex on $a \le x \le b$, then $F(x) = f(a+bx-ax)$ is W_p-convex on $0 \le x \le 1$. Thus by Theorem II

$$(9.5) \qquad \begin{cases} F^{(pk)}(0) = O\left(\lambda_1^{pk}\right) \\ F^{(pk)}(1) = O\left(\lambda_1^{pk}\right) \end{cases} \qquad (k \to \infty) .$$

Hence we have

$$(9.6) \qquad \begin{cases} F^{(pk)}(0) = f^{(pk)}(a)(b-a)^{pk} = O\left(\lambda_1^{pk}\right) \\ F^{(pk)}(1) = f^{(pk)}(b)(b-a)^{pk} = O\left(\lambda_1^{pk}\right) \end{cases} \qquad (k \to \infty) .$$

First, set $a = 0$, $b = x < 1$; then set $a = x > 0$, $b = 1$ to obtain

$$(9.7) \qquad \begin{cases} 0 \le (-1)^k f^{(pk)}(0) x^{pk} \le B\lambda_1^{pk} \\ 0 \le (-1)^k f^{(pk)}(1)(1-x)^{pk} \le B\lambda_1^{pk} \end{cases}$$

which gives (9.4).

Proof of Theorem III.

Using (9.6) we have

$$|f^{(pk)}(x)| \le B\left(\frac{\lambda_1}{b-x}\right)^{pk} \qquad (a \le x \le b) ,$$

where λ_1 is defined by (2.5). Since $b-a > 1$, we choose c such that $b-c > 1$. Thus we have

$$|f^{(pk)}(x)| \le B\left(\frac{\lambda_1}{b-c}\right)^{pk} \qquad (a \le x \le c) .$$

Setting $\frac{1}{b-c} = q$ and applying Lemma 7.1 yields

$$|f^{(pk+j)}(x)| \le Bq^{pk} \qquad (a \le x \le c;\ j=1,2,\ldots,p-1)$$

so that

$$f^{(n)}(x) = 0(q^n) \qquad (n \to \infty)$$

uniformly in $a \le x \le c$. Therefore $f(x)$ is entire of exponential type $q < \lambda_1$. This completes the proof of Theorem III.

10. Minimal W_p-Convex Functions.

In order to obtain necessary and sufficient conditions for a (p,L) series representation we introduce the class of minimal W_p-convex functions (see Definition 3).

Examples of minimal W_p-convex functions are $f(x) = 0$ and $g(x) = M_{p,p-1}(x)$. For the function $g(x)$ choose any $\varepsilon > 0$ and x_o $(0 < x_o < 1)$. Then

$$\left[(-1)^n [M_{p,p-1}(x) - \varepsilon M_{p,p-1}(\lambda_1)]^{(pn)}\right]_{x=x_o} =$$

$$= M_{p,p-1}(x_o) - \varepsilon(\lambda_1)^{pn} M_{p,p-1}(x_o\lambda_1) < 0 ,$$

for sufficiently large n.

Theorem 10.1. *If the series*

$$(10.1)\quad \sum_{n=0}^{\infty} (-1)^n c_{pn} C_{pn}(x) - \sum_{j=0}^{p-2} \frac{M_{p,j}(\lambda_1)}{(\lambda_1)^j} \sum_{n=0}^{\infty} (-1)^n a_{pn+j} A_{pn+j}(x)$$

$$c_{pn} \geq 0 \ ; \quad a_{pn+j} \geq 0 \quad (j = 0,1,\ldots,p-2;\ n = 0,1,\ldots)$$

converges to $f(x)$, *then* $f(x)$ *is a minimal* W_p*-convex function on* $0 \leq x \leq 1$.

Proof. We know from Theorem 9.1 that if (10.1) converges for a single value of x , it converges uniformly in $0 \leq x \leq 1$. Differentiating (10.1) pk times using (4.8) and (4.10) we have

$$(-1)^k f^{(pk)}(x) = \sum_{n=0}^{\infty} (-1)^n c_{p(n+k)} C_{pn}(x) - \sum_{j=0}^{p-2} \frac{M_{p,j}(\lambda_1)}{(\lambda_1)^j} \sum_{n=0}^{\infty} (-1)^n a_{p(n+k)+j} A_{pn+j}(x) \geq 0$$

for $0 \leq x \leq 1$, and from Lemma 2.1

$$(-1)^k f^{(pk+j)}(0) = -\frac{M_{p,j}(\lambda_1)}{(\lambda_1)^j} a_{pn+j} \geq 0 \qquad (j=0,1,\ldots,p-2) .$$

Thus, $f(x)$ is W_p-convex on $0 \leq x \leq 1$. By Lemma 6.2 we have $(-1)^k f^{(pk)}(x) \leq B(\lambda_1)^{pk} T_k$ where

$$T_k = \sum_{n=k}^{\infty} \left[c_{pn} - \sum_{j=0}^{p-2} \frac{M_{p,j}(\lambda_1)}{(\lambda_1)^j} a_{pn+j} \right] \lambda_1^{-pn} .$$

From Theorem 9.1, the power series (9.2) converges absolutely; therefore, $T_k \to 0$ as $k \to \infty$. For a given

$\varepsilon > 0$ and x_o $(0 < x_o < 1)$ there exists an integer k, sufficiently large, such that $BT_k - \varepsilon M_{p,p-1}(x_o\lambda_1) < 0$. In other words $(-1)^k[f(x) - \varepsilon M_{p,p-1}(x\lambda_1)]^{(pk)} < 0$ at $x = x_o$. Hence, $f(x)$ is minimal W_p-convex on $0 \le x \le 1$.

<u>Lemma 10.1</u>. <u>If</u> (i) $f^{(j)}(0) \ge 0$ $(j = 1,2,\ldots,p-2)$, (ii) $f(x) \ge 0$, $-f^{(p)}(x) \ge 0$ <u>for</u> $0 \le x \le 1$ <u>and if</u> (iii) $f(x_o) > \frac{\varepsilon(3p-1)}{(p-1)!}(\lambda_1)^{p-1}$ <u>for some</u> x_o $(0 \le x_o \le 1)$ <u>then</u>

$$f(x) \ge \varepsilon M_{p,p-1}(x\lambda_1) \qquad (0 \le x \le 1). \tag{10.2}$$

<u>Proof</u>. From (7.5) and (iii) of the hypothesis we have

$$f(x) \ge f(x_0)x^{p-1} \ge \frac{\varepsilon(3p-1)}{(p-1)!}(x\lambda_1)^{p-1} \qquad (0 \le x \le x_o). \tag{10.3}$$

From (10.3) it is clear that inequality (10.2) holds for $0 \le x \le x_o$ if we show that

$$\frac{(x\lambda_1)^{p-1}}{(p-1)!} \ge M_{p,p-1}(x\lambda_1) \qquad (0 \le x \le x_o). \tag{10.4}$$

Since

$$M_{p,p-1}(x\lambda_1) = \frac{(x\lambda_1)^{p-1}}{(p-1)!} - \frac{(x\lambda_1)^{2p-1}}{(2p-1)!} + \frac{(x\lambda_1)^{3p-1}}{(3p-1)!} - \cdots$$

inequality (10.4) is equivalent to

$$0 \ge -(x\lambda_1)^{2p-1}t_p(x) \; ; \quad t_p(x) = \frac{1}{(2p-1)!} - \frac{(x\lambda_1)^{3p-1}}{(3p-1)!} + \cdots \tag{10.5}$$

Using (2.6) we have for $0 \le x \le x_o$

$$(10.6) \qquad \frac{1}{(2p-1)!} - \frac{(x\lambda_1)^p}{(3p-1)!} \ge \frac{1}{(2p-1)!}\left[1 - \frac{(2p-1)!(\lambda_1)^p}{(3p-1)!}\right] \ge 0$$

since $1 - \frac{2[(2p-1)!]^2}{(3p-1)!(p-1)!} > 0$, $(p = 2,3,\ldots)$. Then by pairing the terms of the series $t_p(x)$ and using the known estimate (2.6) we see that $t_p(x) > 0$ $(0 \le x \le x_o)$ so that inequality (10.2) holds on that interval. If $x_o = 1$, there is nothing else to show. Therefore, suppose $0 \le x_o < 1$. From (7.6) and (iii) of the hypothesis we have

$$(10.7) \qquad f(x) \ge f(x_o)(1-x^{p-1}) > \frac{\varepsilon(3p-1)(\lambda_1)^{p-1}(1-x^{p-1})}{(p-1)!}$$

$$\ge \frac{\varepsilon(3p-1)(\lambda_1)^{p-1}(1-x)}{(p-1)!} \qquad (x_o \le x \le 1) \quad .$$

To prove the lemma it is enough to show that

$$(10.8) \qquad \frac{(3p-1)(\lambda_1)^{p-1}(1-x)}{(p-1)!} \ge M_{p,p-1}(x\lambda_1) \quad .$$

Equivalently we shall prove that

$$(10.9) \qquad \frac{3p-1}{(p-1)!}(\lambda_1)^{p-1}x \ge M_{p,p-1}(\lambda_1 - x\lambda_1) \qquad (0 \le x \le 1) \quad .$$

Since both sides of (10.9) vanish at $x = 0$ it is sufficient to show that

$$\frac{(3p-1)}{(p-1)!}(\lambda_1)^{p-1} > \max_{0\le x\le 1} [-\lambda_1 M_{p,p-2}(\lambda_1 - x\lambda_1)] = -\lambda_1 M_{p,p-2}(\lambda_1) \ .$$

Now

$$\frac{3p-1}{(p-1)!}(\lambda_1)^{p-1} - \lambda_1 M_{p,p-2}(\lambda_1)$$

$$= \left[\frac{2(2p-1)}{(p-1)!} - \frac{(\lambda_1)^p}{(2p-2)!}\right]\lambda_1^{p-1} + \sum_{n=2}^{\infty}\frac{(\lambda_1)^{pn+p-1}}{(pn+p-2)!}\left[\frac{(pn+p-2)!(\lambda_1)^p}{(pn+2p-2)!} - 1\right] > 0$$

where we use the bounds for $(\lambda_1)^p$ given in (2.6). This completes the proof of Lemma 10.1.

11. Representation of Functions by (p,L) Series.

We now give a sufficient condition for representation of a function by a (p,L) series.

Theorem 11.1. *If $f(x)$ is minimal W_p-convex on $0 \le x \le 1$ then it can be expanded in a convergent (p,L) series.*

Proof. Let

$$(11.1)\quad S_n(x) = \sum_{k=0}^{n}[f^{(pk)}(1)C_{pk}(x) + \sum_{j=0}^{p-2} f^{(pk+j)}(0)A_{pk+j}(x)] .$$

Since $f(x)$ is W_p-convex, we have from Lemma 4.1 that $S_n(x) \le f(x)$ $(0 \le x \le 1;\ n = 0,1,\ldots)$ where $S_n(x)$ is a nondecreasing function of n for each x. Thus $S_n(x) \to g(x)$ (say) as $n \to \infty$. We shall show that $g(x) = f(x)$. For, if $g(x) \ne f(x)$, then for some x_o in $[0,1]$, $f(x_o) - \lim_{n\to\infty} S_n(x_o) = \delta > 0$ and

$$(11.12)\quad f(x_o) - S_n(x_o) = \int_0^1 K_n(x_o,t)f^{(pn)}(t)dt \ge \delta$$

$$(n=1,2,\ldots) .$$

Since $f(x)$ is minimal W_p-convex $f(x) - \varepsilon M_{p,p-1}(x\lambda_1)$ is not W_p-convex for any $\varepsilon > 0$. But we have

$$\left[(-1)^n[f(x) - \varepsilon M_{p,p-1}(x\lambda_1)]^{(pn+j)}\right]_{x=o} = (-1)^n f^{(pn+j)}(0) \geq 0$$

$(j = 0,1,\ldots,p-2;\ n = 0,1,2,\ldots)$. Therefore, choosing $\varepsilon < \dfrac{(p-1)!\delta}{B(3p-1)(\lambda_1)^{p-1}}$ where B is the constant of Lemma 6.4, there exists an integer n_o and an x_o $(0 < x_o < 1)$ such that $(-1)^{n_o} f^{(pn_o)}(x_o) - \varepsilon(\lambda_1)^{pn_o} M_{p,p-1}(x_o\lambda_1) < 0$. Thus, using Lemma 10.1 we have

$$(-1)^{n_o} f^{(pn_o)}(x) \leq \frac{\varepsilon(3p-1)(\lambda_1)^{pn_o+p-1}}{(p-1)!} \qquad (0 \leq x \leq 1).$$

Hence by Lemma 6.4

$$(11.3) \qquad \int_0^1 K_{n_o}(x_o,t) f^{(pn_o)}(t)\,dt \leq \frac{\varepsilon B(3p-1)(\lambda_1)^{p-1}}{(p-1)!} < \delta$$

contradicting (11.2). Thus our assumption that $g(x) \neq f(x)$ is false, which proves the theorem.

Now we are able to prove Theorem IV (§1) which provides us with necessary and sufficient conditions for representation of a function by an absolutely convergent (p,L) series.

<u>Proof of Theorem IV</u>.

(Sufficiency) Let $f(x) = g(x) - h(x)$ where $g(x)$ and $h(x)$ are minimal W_p-convex on $[0,1]$. Thus, by Theorem 11.1

$$g(x) = \sum_{n=0}^{\infty}\left[g^{(pn)}(1)C_{pn}(x) + \sum_{j=0}^{p-2} g^{(pn+j)}(0)A_{pn+j}(x)\right] ,$$

$$h(x) = \sum_{n=0}^{\infty}\left[h^{(pn)}(1)C_{pn}(x) + \sum_{j=0}^{p-2} h^{(pn+j)}(0)A_{pn+j}(x)\right] .$$

Since each series contains only positive terms, their difference is an absolutely convergent (p,L) series whose sum is $f(x)$.

(Necessity) Assume that

$$f(x) = \sum_{n=0}^{\infty}\left[c_{pn}C_{pn}(x) + \sum_{j=0}^{p-2} a_{pn+j}A_{pn+j}(x)\right] \tag{11.4}$$

where the series converges absolutely in the sense that each of the series $\sum_{n=0}^{\infty} c_{pn}C_{pn}(x)$; $\sum_{n=0}^{\infty} a_{pn+j}A_{pn+j}(x)$ $(j = 0,1,...,p-2)$ converges absolutely. Set

$$g(x) = \sum_{n=0}^{\infty}(-1)^n\left[|c_{pn}|C_{pn}(x) + \sum_{j=0}^{p-2}|a_{pn+j}|A_{pn+j}(x)\right] ,$$

$$h(x) = \sum_{n=0}^{\infty}(-1)^n\left[\{|c_{pn}| - (-1)^n c_{pn}\}C_{pn}(x) + \right.$$

$$\left. + \sum_{j=0}^{p-2}\{|a_{pn+j}| - (-1)^n a_{pn+j}\}A_{pn+j}(x)\right] .$$

Both of these series converge since (11.4) converges absolutely and every term of these two series is nonnegative. Thus, by Theorem 10.1, $g(x)$ and $h(x)$ are minimal W_p-convex, and $f(x) = g(x) - h(x)$. This completes the proof of Theorem IV.

12. Conclusion.

Boas [6] has pointed out that Widder's condition (see [38], p. 398) for a real function to be represented by an absolutely convergent Lidstone series, while necessary and sufficient, is not always easy to apply. Here we state, without proof, a generalization of a result of Boas (see [6]; Theorem 1B). This result gives a necessary condition for representation of a function by an absolutely convergent (p,L) series in terms of the growth of the function in the complex plane.

Theorem 12.1. *If the (p,L) series of $f(z)$ converges absolutely to $f(z)$ then*

$$f(z) = o(e^{|z|\lambda_1}) \qquad (|z| \to \infty)$$

where $\lambda_1 \equiv \lambda_1^{(p-1)}$ is defined by (2.5).

The proof depends on two lemmas which are based on the method of Boas and will be given elsewhere.

CHAPTER II

AN ANALOGUE OF COMPLETELY CONVEX FUNCTIONS

1. Introduction.

In this chapter we consider the infinite interpolation problem with periodic interpolation conditions defined by iterating the incidence matrix

$$E_p^3 = \begin{pmatrix} 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & 0 & & 0 & 0 \end{pmatrix} \tag{1.1}$$

with nodes -1, 0 and 1, where p is even. Successive iteration of the matrix (1.1) yields the formal expansion of an entire function $f(z)$ in (p,L^*) series, and we consider the problem of convergence of the series to $f(z)$.

Our object in this chapter is to introduce a class of three-point expansions (called (p,L^*) series) and to obtain some theorems analogous to the results of Chapter I. For $p = 2$, this expansion reduces to the Lidstone series about the points -1 and 1.

In §2 we state Theorem 1.1 and define the fundamental polynomials of the (p,L^*) series. The proof of Theorem 1.1 is given in §3. In §4, we give a relationship between a set of fundamental polynomials of the $(2,L^*)$ series and the Euler polynomials. In §5 we obtain some properties of the zeros of the fundamental polynomials on $[-1,1]$. We obtain in §6 some estimates for the fundamental polynomials of the (p,L^*) series in the interval

$-1 \le x \le 1$. Finally, in §7 we define W_p^*-convex functions and give a sufficient condition for representation of a function by a convergent (p,L^*) series.

2. The (p,L^*) Series.

Let $M_{p,j}(t)$ be the sine function of order p (§2, Chapter I) and let λ_1^* be the smallest positive zero of $M_{p,0}(t)$. Consider the polynomials $Q_{pn}(z)$, $Q_{pn}^*(z)$ and $q_{pn+j}(z)$ $(j = 1,2,\ldots,p-2)$ defined by the following generating functions

$$(2.1) \qquad \sum_{n=0}^{\infty} t^{pn} Q_{pn}(z) = \frac{N_{p,0}(zt)}{N_{p,0}(t)} \equiv \Phi_1(z,t^p)$$

$$(2.2) \qquad \sum_{n=0}^{\infty} t^{pn} Q_{pn}^*(z) = \frac{N_{p,p-1}(zt)}{N_{p,p-1}(t)} \equiv \Phi_2(z,t^p)$$

$$(2.3) \qquad \sum_{n=0}^{\infty} t^{pn+j} q_{pn+j}(z) = t^j \Phi_{j+2}(z,t^p)$$

$$\equiv \begin{cases} N_{p,j}(zt) - \dfrac{N_{p,j}(t)\, N_{p,p-1}(zt)}{N_{p,p-1}(t)} , & j \text{ odd} \\[2ex] N_{p,j}(zt) - \dfrac{N_{p,j}(t)\, N_{p,0}(zt)}{N_{p,0}(t)} , & j \text{ even} \end{cases}$$

$(j = 1,2,\ldots,p-2;\ n = 0,1,\ldots)$.

We now formulate

Theorem 1.1. *Given any even integer* $p \geq 2$, *the following representation holds for every entire function of exponential type* $\tau < \lambda_1^*$:

$$f(z) = \sum_{n=0}^{\infty} f^{(pn)}(-1)q_{pn}(z) + f^{(pn)}(1)q_{pn}(-z) + + \sum_{j=1}^{p-2} f^{(pn+j)}(0)q_{pn+j}(z) \tag{2.4}$$

where $q_{pn}(z) = \frac{1}{2}[Q_{pn}(z) - Q_{pn}^*(z)]$; $q_{pn}(-z) = \frac{1}{2}[Q_{pn}(z) + Q_{pn}^*(z)]$ *and the polynomials* $\{Q_{pn}(z)\}$, $\{Q_{pn}^*(z)\}$ *and* $\{q_{pn+j}(z)\}_{n=0}^{\infty}$ $(j = 1,2,...,p-2)$ *are given by* (2.1), (2.2) *and* (2.3). *The series on the right in* (2.4) *converges to* $f(z)$ *for all* z *and the convergence is uniform in all bounded subsets of the plane*.

This theorem leads to the following

Definition 1. *Let* $p \geq 2$ *be an even integer. We shall say that the series* (2.4) *is the* (p,L^*) *series of* f *and that* $\{q_{pn+j}(z)\}_{n=0}^{\infty}$ $(j = 0,1,...,p-2)$ *are the fundamental polynomials of the* (p,L^*) *series*.

We observe on comparing (2.2) and (2.3) with (2.11) and (2.12) of Chapter I that

$$q_{pn+j}(z) \equiv A_{pn+j}(z) \qquad (j=1,3,...,p-3) \tag{2.5}$$

$$Q_{pn}^*(z) \equiv C_{pn}(z) \tag{2.6}$$

3. Proof of Theorem 1.1.

Setting $f(z) = e^{zt}$ in (2.4), we get the formal (p,L^*) series representation of e^{zt}, so that

$$(3.1)\quad e^{zt} = \Phi_1(z,t^p)\cosh t + \Phi_2(z,t^p)\sinh t + \sum_{j=1}^{p-2} t^j \Phi_{j+2}(x,t^p) .$$

Replacing t successively in this relation by ωt, $\omega^2 t$, $\ldots$, $\omega^{p-1} t$ with $\omega = e^{2\pi i/p}$, p even, and observing that $\Phi_j(z,t^p)$ $(j = 0,1,\ldots,p-2)$ remains unchanged, we get the following system of equations in Φ_j :

$$(3.2)\quad e^{\omega^m zt} = \Phi_1 \cosh \omega^m t + \Phi_2 \sinh \omega^m t + \sum_{j=1}^{p-2} (\omega^m t)^j \, \Phi_{j+2} \qquad (m=0,1,\ldots,p-1) .$$

Using (3.3) and (3.4), Chapter I, we obtain (2.1) and (2.2) from (3.2). To solve for $\Phi_{j+2}(z,t^p)$ $(j = 1,2,\ldots,p-2)$, we multiply the $(m+1)^{th}$ equation of (3.2) by ω^{-mj} and add. Using (3.3) and (3.4), Chapter I, we obtain the easily verified identities, for p even :

$$(3.5)\quad \sum_{s=0}^{p-1} \omega^{(p-j)s} \cosh \omega^s t = \begin{cases} pN_{p,j}(t) & , j \text{ even} \\ 0 & , j \text{ odd} \end{cases}$$

$$(3.6)\quad \sum_{s=0}^{p-1} \omega^{(p-j)s} \sinh \omega^s t = \begin{cases} 0 & , j \text{ even} \\ pN_{p,j}(t) & , j \text{ odd} \end{cases}$$

Therefore, we have (2.3) for $j = 1,2,\ldots,p-2$.

The right hand side of (3.1) is regular except for the simple poles at the zeros of the functions $N_{p,0}(t)$ and $N_{p,p-1}(t)$. Thus from Lemma 2.1, Chapter I, the generating functions (2.1), (2.2) and (2.3) converge

uniformly in (at least) any compact subset of $|t| < \lambda_1^*$ and the expansion

$$e^{zt} = (\cosh t) \sum_{n=0}^{\infty} t^{pn} Q_{pn}(z) + (\sinh t) \sum_{n=0}^{\infty} t^{pn} Q_{pn}^*(z) +$$

$$+ \sum_{j=1}^{p-2} \sum_{n=0}^{\infty} t^{pn+j} q_{pn+j}(z) \quad , \quad p \text{ even}$$

is valid for $|t| < \lambda_1^*$. The proof is completed by kernel expansion method (see [7], p. 10) with e^{zt} as kernel.

<u>Remarks</u>

(1) Theorem 1.1 yields a "best possible" result. Consider the function $M_{p,0}(x\lambda_1^*)$ where $M_{p,0}(x)$ is defined by (2.1), Chapter I, and λ_1^* is defined by (2.5), Chapter I. $M_{p,0}(x\lambda_1^*)$ is a real entire function of exponential type λ_1^* whose (p,L^*) series expansion (2.4) is identically zero. Thus, the upper bound λ_1^* on the exponential type in Theorem 1.1 cannot be replaced by any larger number.

(2) It is easy to see from the behaviour of the real zeros of $M_{p,j}(t)$ $(j = 0,1,\ldots,p-1)$(see Lemma 2.1, Chapter I) that, roughly speaking, for "large" p, the class of entire functions of exponential type having a valid (p,L^*) series is also "large".

4. The Polynomials $Q_{4n}(z)$ and the Euler Polynomials.

We consider here the expansion of functions about the three points -1, 0 and 1 defined formally by (2.4), in the particular case where $p = 4$. It is our object in this section to show that there exists a relationship between the polynomials $Q_{4n}(z)$ defined by setting $p = 4$ in (2.1) and the Euler polynomials $E_n(z)$ which are given by

$$\frac{2e^{zt}}{e^t+1} = \sum_{n=0}^{\infty} \frac{t^n E_n(z)}{n!} \tag{4.1}$$

Theorem 4.1. *For* $n = 0,1,2,\ldots$ *we have*

$$Q_{4n}(z) = \frac{(-4)^n}{(4n)!} \sum_{k=0}^{2n} \binom{4n}{2k} (-1)^k E_{2k}\left(\frac{z+1}{2}\right) E_{4n-2k}\left(\frac{z+1}{2}\right) , \tag{4.2}$$

where the polynomials $Q_{4n}(z)$ *are defined by* (2.1). *Also*

$$E_{4n}(z) + E_{4n}(z+1) = 2 \sum_{k=0}^{n} \binom{4n}{4k} (4k)!\, Q_{4k}(z) . \tag{4.2a}$$

Proof of Theorem 4.1.

From (4.1) we have

$$\frac{2e^{zt}}{e^t+1} = \frac{2e^{zt}e^{-\frac{t}{2}}}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}} = \frac{2e^{(z-\frac{1}{2})t}}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}} = \sum_{n=0}^{\infty} \frac{E_n(z)t^n}{n!} . \tag{4.3}$$

Replacing z by $1-z$ in (4.3) and adding we have

$$(4.4) \qquad \frac{2\cosh\,(z-\frac{1}{2})t}{\cosh\frac{t}{2}} = \sum_{n=0}^{\infty} [E_n(z) + E_n(1-z)]\frac{t^n}{n!} \quad .$$

Replacing t by $2t$ and $2z-1$ by z in (4.4), and observing that $E_n(1-z) = (-1)^n E_n(z)$ yields

$$(4.5) \qquad \frac{\cosh\,zt}{\cosh\,t} = \sum_{n=0}^{\infty} E_{2n}\left(\frac{z+1}{2}\right)\frac{(2t)^{2n}}{(2n!)} \quad .$$

Replacing t by it we have

$$(4.6) \qquad \frac{\cos\,zt}{\cos\,t} = \sum_{n=0}^{\infty} E_{2n}\left(\frac{z+1}{2}\right)\frac{(-1)^n(2t)^{2n}}{(2n)!} \quad .$$

Finally, replacing t by $\left(\frac{1+i}{2}\right)t$ and $\left(\frac{1-i}{2}\right)t$ respectively in (4.6), and multiplying, we have

$$(4.7) \qquad \frac{\cos\left(\frac{1+i}{2}\right)zt\,\cos\left(\frac{1-i}{2}\right)zt}{\cos\left(\frac{1+i}{2}\right)t\,\cos\left(\frac{1-i}{2}\right)t}$$

$$= \sum_{n=0}^{\infty}\frac{t^{4n}}{(4n)!}(-4)^n \sum_{n=0}^{2n}\binom{4n}{2k}(-1)^k E_{2k}\left(\frac{1+z}{2}\right)E_{4n-2k}\left(\frac{1+z}{2}\right) \quad .$$

Setting $p = 4$ in (2.1) it is easily seen that

$$(4.8) \qquad \sum_{n=0}^{\infty} t^{4n}Q_{4n}(z) = \frac{\cos\left(\frac{1+i}{2}\right)zt\,\cos\left(\frac{1-i}{2}\right)zt}{\cos\left(\frac{1+i}{2}\right)t\,\cos\left(\frac{1-i}{2}\right)t}$$

The proof of (4.2) is completed by comparing the coefficient of t^{4n} in (4.7) and (4.8). To prove (4.2a) we use (2.1) and the known identity $2z^n = E_n(z) + E_n(z+1)$ which is easily verified from (4.1).

5. The Properties of Zeros of the Fundamental Polynomials of (p, L^*) Series.

Suppose we want to define a class of functions which belong to $C^\infty[-1,1]$ and have the additional property that each term of the right hand side of the formal expansion (2.4) is nonnegative. Recall that from the (p,L) series expansions of Chapter I, we defined the classes of W_p-Convex functions.

In an attempt to obtain analogous classes of functions for the three-point case considered in this chapter, we require the following. Let L be the linear operator defined on $C^p[-1,1]$ by

$$(5.1)\quad L(f) = f(x) - \left[f(-1)q_0(x) + f(1)q_0(-x) + \sum_{j=1}^{p-2} f^{(j)}(0)q_j(x)\right]$$

where $q_j(x)$ $(j = 0,1,\ldots,p-2)$ are defined by (2.1), (2.2), and (2.3).

Since $L(P) = 0$ for any polynomial $P(x)$ of degree $\leq p-1$, we have by Peano's theorem [12]

$$(5.2)\qquad L(f) = \int_{-1}^{1} H_1(x,t)f^{(p)}(t)dt$$

$$(5.3)\qquad (p-1)!H_1(x,t) = L_x[(x-t)_+^{p-1}] \quad .$$

Setting $f(x) = 1, x, \ldots, x^{p-1}$ successively in (5.1), we

easily have

$$(5.4)\quad \begin{cases} 2q_0(z) = 1-z^{p-1} \quad ; \quad (2j)!\,q_{2j}(z) = z^{2j}-1 \\ (2j-1)!\,q_{2j-1}(z) = z^{2j-1}-z^{p-1} \quad (j=1,2,\ldots,\frac{p}{2}),\ p \text{ even} \end{cases}$$

Now $H_1(x,t)$ is the Green's function for the differential system

$$(5.5)\quad \begin{cases} f^{(p)}(x) = \phi(x) \\ f(-1) = f(1) = 0 \ ; \quad f^{(j)}(0) = 0 \quad (j=1,\ldots,p-2) \end{cases}$$

where $\phi(x)$ is any function continuous in $-1 \le x \le 1$. That is to say, the unique solution of the system (5.5) for a given $\phi(x)$ is

$$(5.6)\qquad f(x) = \int_{-1}^{1} H_1(x,t)\phi(t)dt \quad .$$

Define

$$(5.7)\qquad H_n(x,t) = \int_{-1}^{1} H_1(x,u)H_{n-1}(u,t)du \quad (n=2,3,\ldots) ,$$

then we have

<u>Lemma 5.1</u>. <u>The following inequalities hold</u>:

$$(5.8)\qquad (-1)^n H_n(x,t) \ge 0 \qquad (-1\le x\le 1;\ -1\le t\le 1;\ n=1,2,\ldots)$$

$$(5.9)\qquad (-1)^n q_{pn}(x) \ge 0 \qquad (-1\le x\le 1;\ n=0,1,\ldots)$$

$$(5.10)\qquad (-1)^{n+j} q_{pn+j}(-x) = (-1)^n q_{pn+j}(x) \ge 0$$

$$(0\le x\le 1;\ j=1,2,\ldots,p-2;\ n=0,1,\ldots) .$$

<u>Proof</u>. (5.8) is easily verified for $n = 1$ from (5.1) and (5.3). Then for $n = 2,3,\ldots$, (5.8) follows from (5.7). Since the polynomials $q_{pn+j}(x)$ $(j = 0,1,\ldots,p-2;\ n = 1,2,\ldots)$ satisfy the differential system (5.5) with $\phi(x) = q_{p(n-1)+j}(x)$ we have, using (5.7)

$$q_{pn+j}(x) = \int_{-1}^{1} H_n(x,t)q_j(t)dt \quad (j=0,1,\ldots,p-2) . \tag{5.11}$$

Then (5.9) and (5.10) follow from (5.4), (5.8) and (5.11). This proves the lemma.

<u>Lemma 5.2</u>. <u>If</u> $f(x)$ <u>belongs to</u> $C^{pn}[-1,1]$, <u>then</u>

$$\begin{aligned} f(x) = \sum_{k=0}^{n-1} f^{(pn)}(-1)q_{pn}(x) + f^{(pn)}(1)q_{pn}(-x) + \\ + \sum_{j=1}^{p-2} f^{(pn+j)}(0)q_{pn+j}(x) + R_n(x,f) \end{aligned} \tag{5.12}$$

<u>where</u>

$$R_n(x,f) = \int_{-1}^{1} H_n(x,t)f^{(pn)}(t)dt \tag{5.13}$$

<u>and</u> $H_n(x,t)$ <u>is given by</u> (5.3) <u>and</u> (5.7).

<u>Proof</u>. From (5.1) and (5.2) we get (5.12) for $n = 1$. The proof is completed by induction on n.

6. Estimates on the Fundamental Polynomials of the (p,L^*) Series.

We shall use methods similar to those of Chapter I, §6, to obtain estimates for the fundamental polynomials of the (p,L^*) series in the interval $[-1,1]$. We let B denote suitable constants (not necessarily the same) which are independent of n and x $(-1 \le x \le 1)$ unless otherwise stated.

Lemma 6.1. *For* $-1 \le x \le 1$; $n = 0,1,\ldots$, *and* p *even, we have*

$$(6.1)\quad \left|(-1)^n Q_{pn}(x) - \frac{pM_{p,0}(x\lambda_1^*)}{M_{p,p-2}(\lambda_1^*)(\lambda_1^*)^{pn+1}}\right| < \frac{B}{(r_1^*)^{pn+1}}$$

$$(6.2)\quad \left|(-1)^{n+1} Q^*_{pn}(x) - \frac{pM_{p,p-1}(x\lambda_1)}{M_{p,p-2}(\lambda_1)(\lambda_1)^{pn+1}}\right| < \frac{B}{(\lambda_2)^{pn+1}}$$

$$(6.3)\quad \left|(-1)^n q_{pn+j}(x) - \frac{pM_{p,j}(\lambda_1)\, M_{p,p-1}(x\lambda_1)}{M_{p,p-2}(\lambda_1)(\lambda_1)^{pn+j+1}}\right| < \frac{B}{(r_1)^{pn+1}}$$

$$(j=1,3,\ldots,p-3)$$

$$(6.4)\quad \left|(-1)^{n+1} q_{pn+j}(x) - \frac{pM_{p,j}(\lambda_1^*)\, M_{p,0}(x\lambda_1^*)}{M_{p,p-1}(\lambda_1^*)(\lambda_1^*)^{pn+j+1}}\right| < \frac{B}{(r_1^*)^{pn+1}}$$

$$(j=2,4,\ldots,p-2)\ ,$$

where $r_1 = \frac{1}{2}(\lambda_1 + \lambda_2)$; $r_1^* = \frac{1}{2}(\lambda_1^* + \lambda_2^*)$ *and* λ_k *and* λ_k^* *are given by* (2.5), *Chapter I*.

Proof. (6.2) and (6.3) follow from (2.5), (2.6), (5.10) and (6.1), (6.2), Chapter I, by observing that $Q^*_{pn}(-x) = -Q^*_{pn}(x)$. (6.1) and (6.4) are proved by the same techniques used to prove Lemma 6.1, Chapter I.

Lemma 6.2. There exist constants B such that for $-1 \le x \le 1$, $n = 0,1,\ldots$, and p even

$$0 \le (-1)^n Q_{pn}(x) \le \frac{B}{(\lambda_1^*)^{pn}} \tag{6.5}$$

$$0 \le (-1)^{n+1} q_{pn+j}(x) \le \frac{B}{(\lambda_1^*)^{pn}} \qquad (j=2,4,\ldots,p-2) \tag{6.6}$$

$$\left| Q^*_{pn}(x) \right| \le \frac{B}{(\lambda_1)^{pn}} \tag{6.7}$$

$$\left| q_{pn+j}(x) \right| \le \frac{B}{(\lambda_1)^{pn}} \qquad (j=1,3,\ldots,p-3) \tag{6.8}$$

$$0 \le (-1)^n q_{pn}(x) \le \frac{B}{(\lambda_1^*)^{pn}} \tag{6.9}$$

where λ_1 and λ_1^* are defined by (2.5), Chapter I.

Proof. (6.7) and (6.8) follow from (2.15), (2.16), and (6.8), (6.9), Chapter I. From (6.1) we have

$$0 \le (-1)^n Q_{pn}(x) \le \frac{B}{(r_1^*)^{pn+1}} + \left| \frac{pM_{p,0}(x\lambda_1^*)}{M_{p,p-1}(\lambda_1^*)(\lambda_1^*)^{pn+1}} \right| \le \frac{B}{(\lambda_1^*)^{pn}}$$

since $M_{p,0}(x\lambda_1^*)$ is uniformly bounded in $[-1,1]$. (6.6) is proved similarly from (6.4). (6.9) follows from (6.5), (6.7) and the observation that $\lambda_1^* < \lambda_1$.

Lemma 6.3. *For any fixed* x_o, $(0 < x_o < 1)$ *there exist constants* B *such that for* p *even*

(6.10) $$(-1)^n Q_{pn}(x_o) > \frac{B}{(\lambda_1^*)^{pn}}$$

(6.11) $$(-1)^n Q_{pn}^*(x_o) > \frac{B}{(\lambda_1)^{pn}}$$

(6.12) $$(-1)^n q_{pn+j}(x_o) > \frac{B}{(\lambda_1)^{pn}} \qquad (j=1,3,\ldots,p-3)$$

(6.13) $$(-1)^{n+1} q_{pn+j}(x_o) > \frac{B}{(\lambda_1^*)^{pn}} \qquad (j=2,4,\ldots,p-2) .$$

Also we have

(6.14) $$(-1)^n q_{pn}(x_o) > \frac{B}{(\lambda_1^*)^{pn}} \qquad (-1<x_o<1) .$$

Proof. We shall prove (6.10). From (6.1) we have

$$\lim_{n\to\infty} \frac{(-1)^n Q_{pn}(x_o)\ M_{p,p-2}(\lambda_1^*)(\lambda_1^*)^{pn}}{M_{p,0}(x_o\lambda_1^*)} = \frac{p}{\lambda_1^*}$$

from which (6.10) easily follows. (6.11) to (6.14) are proved in an analogous way.

Lemma 6.4. *For* $-1 \le x \le 1$, $n = 1,2,\ldots$, *and* p *even*

(6.15) $$0 \le (-1)^n \int_{-1}^{1} H_n(x,t)\,dt \le \frac{B}{(\lambda_1^*)^{pn}} ,$$

where $H_n(x,t)$ *is defined by* (5.3) *and* (5.7).

Proof. Since $Q_0(x) \equiv 1$ we have

$$0 \le (-1)^n \int_{-1}^{1} H_n(x,t)\,dt = (-1)^n Q_{pn}(x)$$

7. (p,L^*) Series and W_p^*-Convex Functions.

Theorem 7.1. If the series

$$a_0 Q_0(x) + a_0^* Q_0^*(x) + a_1 q_1(x) + \dots + \tag{7.1}$$

$$+ a_{p-2} q_{p-2}(x) + a_p Q_p(x) + \dots \quad , \quad p \text{ even}$$

converges for a single value $x_o \ne 0$ $(-1 < x_o < 1)$ then it converges uniformly in $-1 \le x \le 1$ to a function $f(x)$. Furthermore, the series

$$\sum_{n=0}^{\infty} \left\{ \frac{(-1)^n}{(\lambda_1)^{pn}} \left[a_{pn}^* - \sum_{\substack{j=1 \\ (j \text{ odd})}}^{p-3} \frac{M_{p,j}(\lambda_1) a_{pn+j}}{(\lambda_1)^j} \right] + \frac{(-1)^n}{(\lambda_1^*)^{pn}} \left[a_{pn} - \sum_{\substack{j=2 \\ (j \text{ even})}}^{p-2} \frac{M_{p,j}(\lambda_1^*) a_{pn+j}}{(\lambda_1^*)^j} \right] \right\} \tag{7.2}$$

converges and we have

$$f^{(pk)}(x) = a_{pk} Q_0(x) + a_{pk}^* Q_0^*(x) + a_{pk+1} q_1(x) + \dots \tag{7.3}$$

$$+ a_{pk+p-2} q_{p-2}(x) + a_{p(k+1)} Q_p(x) + \dots$$

for $-1 \le x \le 1$

Proof. If (7.1) converges for $x = x_o \neq 0$ then

$$\lim_{n\to\infty} a^*_{pn} Q^*_{pn}(x_o) = 0 \quad ; \quad \lim_{n\to\infty} a_{pn} Q_{pn}(x_o) = 0 \quad ;$$

$$\lim_{n\to\infty} a_{pn+j} q_{pn+j}(x_o) = 0 \qquad (j = 1,2,\ldots,p-2) .$$

So, by Lemma 6.3, we have $a_{pn} = 0\left(\lambda_1^{*\ pn}\right)$; $a^*_{pn} = 0\left|\lambda_1{}^{pn}\right|$

$a_{pn+j} = 0\left|\lambda_1{}^{pn}\right|$ $(j = 1,2,\ldots,p-3)$; $a_{pn+j} = 0\left(\lambda_1^{*\ pn}\right)$ $(j = 2,4,\ldots,p-2)$.

With suitable modifications, the proof follows Widder's method (see [38]; Theorem 5.2, p. 392).

Definition 7.1. *A real function is said to be W^*_p-convex, p even, on the interval $a \le x \le b$ if*

(i) $f \in C^\infty[a,b]$

(ii) $(-1)^k f^{(pk)}(x) \ge 0 \qquad (a \le x \le b;\ k=0,1,\ldots)$

(iii) $(-1)^{k+1} f^{(pk+2j)}\left(\frac{a+b}{2}\right) \ge 0 \qquad (j=1,2,\ldots,\frac{p}{2}-1;\ k=0,1,\ldots)$

(iv) $f^{(pk+2j-1)}\left(\frac{a+b}{2}\right) = 0 \qquad (j=1,2,\ldots,\frac{p}{2}-1;\ k=0,1,\ldots)$.

For p even, the function $M_{p,0}(x\lambda_1^*)$ is W^*_p-convex on $-1 \le x \le 1$, where λ_1^* is the smallest positive zero of $M_{p,0}(x)$. We now give some results on W^*_p-convex functions.

Lemma 7.1. *If $f(x)$ is W^*_p-convex in $-1 \le x \le 1$, then*

$$\left\{\begin{aligned} f^{(pk)}(-1) &= 0\left(\lambda_1^{*\ pk}\right) \\ f^{(pk)}(1) &= 0\left(\lambda_1^{*\ pk}\right) \end{aligned}\right. \qquad (k \to \infty) . \tag{7.4}$$

<u>Proof</u>. From Definition 7.1 and (5.12) every term in the (p,L^*) series of $f(x)$ is nonnegative and we have

$$(7.5)\qquad \begin{cases} 0 \le f^{(pk)}(-1)q_{pk}(x) \le f(x) \\ 0 \le f^{(pk)}(1)q_{pk}(-x) \le f(x) \end{cases}$$

Take $x = 0$ and apply (6.14). Then we have (7.4).

<u>Lemma 7.2</u>. <u>If</u> $f(x)$ <u>is</u> W_p^*-<u>convex</u> <u>in</u> $-1 \le x \le 1$, <u>then</u> <u>there is a constant</u> B <u>such that</u>

$$(7.6)\qquad \begin{cases} 0 \le (-1)^k f^{(pk)}(x) \le B\left(\dfrac{2\lambda_1^*}{1+x}\right)^{pk} \\ 0 \le (-1)^k f^{(pk)}(x) \le B\left(\dfrac{2\lambda_1^*}{1-x}\right)^{pk} , \qquad (k \to \infty) . \end{cases}$$

<u>Proof</u>. If $f(x)$ is W_p^*-convex in $a \le x \le b$, then $F(x) = f\left[\left(\frac{b-a}{2}\right)x + \left(\frac{a+b}{2}\right)\right]$ is W_p^*-convex in $-1 \le x \le 1$. By Lemma 7.1 we have

$$(7.7)\qquad \begin{cases} F^{(pk)}(-1) = \left(\dfrac{b-a}{2}\right)^{pk} f^{(pk)}(a) = O\left(\lambda_1^{*pk}\right) \\ F^{(pk)}(1) = \left(\dfrac{b-a}{2}\right)^{pk} f^{(pk)}(b) = O\left(\lambda_1^{*pk}\right) , \quad (k \to \infty). \end{cases}$$

First set $a = -1$, $b = x < 1$; then set $a = x > -1$, $b = 1$ to obtain (7.6). From (7.5) it is clear that B is independent of x , $(-1 \le x \le 1)$ in (7.6), and the lemma is proved.

Now we have

<u>Theorem 7.2</u>. <u>If</u> $f(x)$ <u>is</u> W_p^*-<u>convex in</u> $a \le x \le b$ <u>with</u> $b-a > 2$, <u>then</u> $f(x)$ <u>is entire of exponential type less than</u> λ_1^* <u>and the</u> (p,L^*) <u>series representation holds for all</u> z <u>in the complex plane</u>.

<u>Proof</u>. Using the modifications provided by the previous two lemmas, the proof follows Widder's method (see [38]; Theorem 6.3, p. 395).

Theorem 7.2 gives a sufficient condition for representation of a function by a convergent (p,L^*) series. The function $N_{p,0}(x)$ is not W_p^*-convex but has the (p,L^*) series representation $N_{p,0}(x) = N_{p,0}(1) \sum_{n=0}^{\infty} Q_{pn}(x)$. This example shows that Theorem 7.2 does not provide necessary conditions for representation by a (p,L^*) series.

A class of minimal W_p^*-convex functions can be defined in order to obtain necessary and sufficient conditions for representation by a (p,L^*) series, but for want of complete results, we do not discuss them here.

CHAPTER III

LACUNARY INTERPOLATION (0,n-1,n) CASE

1. Introduction.

If E is a set of n real distinct points, we shall be concerned with the problem of finding the explicit form of the unique polynomial $P(x)$ of degree $\le 3n-1$, when the values of $P(x)$, $P^{(n-1)}(x)$ and $P^{(n)}(x)$ are assigned on E. We shall call this the problem of $(0,n-1,n)$ interpolation on E. The existence and uniqueness of these polynomials is a special case of a general result of Atkinson and Sharma [1] (see also [14] and [27]).

In §2 we deal with notations and the statement of the main theorems, and we show by an example that $(0,n-1,n)$ interpolation is not always possible when E contains complex points. In this connection we may observe that as a special case of a theorem of Ferguson [14], it follows that the problem of $(0,n-1,n)$ interpolation is not always uniquely solvable when E is allowed to contain points from the complex plane. However, for the roots of unity, the problem has a unique solution. For relevant literature on this type of problem we refer to the work of Surányi and Turán [29], O. Kiš [15], and Sharma [26].

In §3 we give the proofs of Theorems 1 and 2. §4 deals with estimates on the fundamental polynomials defined by (2.21), (2.22) and (2.23) when E consists of

the n^{th} roots of unity which lead to the solution of a convergence problem. Also, in §4 we state and prove Theorem 4 which shows that the Dini-Lipschitz condition of Theorem 3 cannot be relaxed. In §5 we prove Theorem 5, a result on least squares convergence. Theorems 4 and 5 are analogous to known results ([28] and [32]) for Lagrange interpolation, pointing out the similarity of behaviour of $(0,n-1,n)$ interpolation polynomials and Lagrange interpolation polynomials for large values of n.

2. Preliminaries and Statements of Theorems.

If E is a set of n real points

$$x_1 < x_2 < \dots < x_n \tag{2.1}$$

set $w(x) = \prod_{j=1}^{n}(x-x_j)$ and let $\ell_k(x) = \dfrac{w(x)}{(x-x_k)w'(x_k)}$, $(k = 1,2,\dots,n)$ denote the fundamental polynomials of Lagrange interpolation. Then set

$$Q(x) = \frac{1}{(n-2)!}\int_0^1 w^2(t)(x-t)^{n-2}dt \tag{2.2}$$

$$\Omega_k(x) = \frac{1}{(n-2)!}\int_0^x \ell_k^2(t)\left[1 - \frac{w''(x_k)}{w'(x_k)}(t-x_k)\right](x-t)^{n-2}dt \tag{2.3}$$

$$\tau_k(x) = \frac{1}{(n-2)!}\int_0^1 \ell_k^2(t)(t-x_k)(x-t)^{n-2}dt \tag{2.4}$$

$$W(x) = \frac{Q(x) - L_n(Q;x)}{[x_1,\dots,x_n;Q]} \tag{2.5}$$

where

$$L_n(Q;x) = \sum_{k=1}^{n} Q(x_k)\ell_k(x) \quad \text{and} \quad [x_1,\ldots,x_n;f] \equiv \sum_{k=1}^{n} f(x_k)/w'(x_k)$$

denotes the divided difference of f on the nodes (2.1). The definition (2.5) is justified because of the following.

__Lemma 2.1__. __For__ n __real nodes__ (2.1), __we have__

$$[x_1,\ldots,x_n;Q] > 0 \ .$$

__Proof__. Here we shall use the following inequality which is a particular case of a result of Curry and Schoenberg ([11], Theorem 1, p. 74).

$$[x_1,\ldots,x_n;(x-t)_+^{n-2}] > 0 \qquad (x_1 \le t \le x_n) \tag{2.6}$$

where

$$(x-t)_+^{\nu} = \begin{cases} (x-t)^{\nu}, & t < x \\ 0, & t \ge x \end{cases} \ .$$

Since

$$Q(x) = \int_0^{x_1} w^2(t)(x-t)^{n-2}dt + \int_{x_1}^{x_n} w^2(t)(x-t)_+^{n-2}dt$$

$$\equiv A(x) + \tilde{Q}(x) \quad ,$$

and since $[x_1,\ldots,x_n;A(x)] = 0$ because $A(x)$ is a polynomial of degree $\le n-2$, we have

$$[x_1,\ldots,x_n;Q] = [x_1,\ldots,x_n;\tilde{Q}]$$

$$= \int_{x_1}^{x_n} w^2(t)[x_1,\ldots,x_n;(x-t)_+^{n-2}]dt$$

which proves the lemma on using (2.6).

If $\{\alpha_k\}_1^n$, $\{\beta_k\}_1^n$, $\{\gamma_k\}_1^n$ are three given sets of real or complex numbers, then the polynomials $\Pi_n(x)$ of degree $\leq 3n-1$, having the properties

$$(2.7)\qquad \Pi_n(x_k) = \alpha_k \ , \ \Pi_n^{(n-1)}(x_k) = \beta_k \ , \ \Pi_n^{(n)}(x_k) = \gamma_k \ (k=1,\ldots,n) \ ,$$

have the form

$$(2.8)\qquad \Pi_n(x) = \sum_1^n \alpha_k R_k(x) + \sum_1^n \beta_k S_k(x) + \sum_1^n \gamma_k T_k(x)$$

where $R_k(x)$, $S_k(x)$, $T_k(x)$ are the <u>fundamental polynomials</u> of this interpolation problem and are determined by the following properties:

$$(2.9)\qquad R_k(x_j) = \delta_{jk} \ , \ R_k^{(n-1)}(x_j) = R_k^{(n)}(x_j) = 0$$

$$(2.10)\qquad S_k(x_j) = S_k^{(n)}(x_j) = 0 \ , \ S_k^{(n-1)}(x_j) = \delta_{jk}$$

$$(2.11)\qquad T_k(x_j) = T_k^{(n-1)}(x_j) = 0 \ , \ T_k^{(n)}(x_j) = \delta_{jk}$$

where δ_{jk} is the Kronecker delta. We now formulate

Theorem 1. *For given real nodes* (2.1), *the fundamental polynomials of interpolation satisfying* (2.9), (2.10) *and* (2.11) *respectively have the following explicit representations*:

(2.12) $$R_k(x) = \ell_k(x) + \frac{W(x)}{w'(x_k)}$$

(2.13) $$S_k(x) = \Omega_k(x) - L_n(\Omega_n;x) - [x_1,\ldots,x_n;\Omega_k]\ W(x)$$

(2.14) $$T_k(x) = \tau_k(x) - L_n(\tau_k;x) - [x_1,\ldots,x_n;\tau_k]\ W(x)$$

where $W(x)$, $\Omega_k(x)$ *and* $\tau_k(x)$ *are given by* (2.3), (2.4) *and* (2.5).

When the nodes (2.1) are taken to be complex numbers $z_1,\ldots,z_n$ then in general $[z_1,\ldots,z_n;Q]$ may vanish as is easily verified on taking $n = 3$, $z_2 = -1$, $z_3 = 1$. Indeed we see by easy computation that

$$[z_1,-1,1;Q(z)] =$$

$$= \int_0^1 (1-t)^3\{t^4z_1^6 + (t^4-2t^2)z_1^4 + (t^2-1)^2\ z_1^2 + t^2(t+1)^2\}dt =$$

$$= 3z_1^6 - 25z_1^4 + 185z_1^2 + 23$$

which has no real zeros, as can be easily verified.

However, when $\{z_k\}_1^n$ are the n^{th} roots of unity, with

(2.15) $$z_k = e^{2k\pi i/n} \qquad (k=0,1,\ldots,n-1)$$

we have,

$$(2.16)\quad \ell_k(z) = \left(\frac{z^n-1}{z-z_k}\right)\frac{z_k}{n} \quad ; \quad Q(z) = \frac{z^{n-1}}{(n-2)!}\int_0^1 (1-z^n t^n)^2 (1-t)^{n-2} dt$$

so that using (2.2) to (2.5) we get

$$(2.17)\qquad [z_1,\ldots,z_n;Q] = Q(1) > 0$$

$$(2.18)\qquad L_n(Q;z) = z^{n-1}Q(1)$$

$$(2.19)\qquad W(z) = z^{n-1}\left[\frac{Q(z)}{Q(1)} - 1\right]$$

where $Q(z)$ is given by (2.16). Further, we get

$$(2.20)\qquad \Omega_k(z) = \frac{z^{n-1}}{(n-2)!}\int_0^1 \ell_k^2(zt)\left[1 - \frac{n-1}{z_k}(zt-z_k)\right](1-t)^{n-2}dt$$

$$(2.21)\qquad \tau_k(z) = \frac{z^{n-1}}{(n-2)!}\int_0^1 (zt-z_k)\ell_k^2(zt)(1-t)^{n-2}dt \quad .$$

We now formulate

<u>Theorem 2</u>. <u>If</u> $z_k = e^{2\pi ki/n}$, $(k = 1,\ldots,n)$ <u>then the fundamental polynomials of</u> $(0,n-1,n)$ <u>interpolation are given by</u> $R_k(z)$, $S_k(z)$, $T_k(z)$ $(k = 1,\ldots,n)$ <u>where</u>

$$(2.22)\qquad R_k(z) = \ell_k(z) + \frac{z_k}{n}W(z)$$

$$(2.23)\qquad S_k(z) = \sum_{j=1}^{n}\ell_j(z)\left[\Omega_k(z) - \frac{\Omega_{\lambda(k,j)}(1)}{z_j}\right] - \frac{W(z)}{n!}$$

$$(2.24)\qquad T_k(z) = \sum_{j=1}^{n}\ell_j(z)\left[\tau_k(z)-\tau_{\lambda(k,j)}(1)\right] + $$

$$+ \left(1-\frac{1}{n}\right)\frac{W(z)}{n!}\int_0^1 t(1-t^n)(1-t)^{n-2}dt$$

<u>where</u> $\lambda(k,j)$ <u>is a positive integer</u> $\leq n$ <u>such that</u> $\lambda(k,j) \equiv n+k-j \pmod{n}$.

<u>Theorem 3</u>. <u>If</u> $f(z)$ <u>is analytic in</u> $|z| < 1$ <u>and continuous for</u> $|z| = 1$, <u>let</u> $\omega(\delta)$ <u>be the modulus of continuity of</u> $f(e^{i\theta})$, $0 \leq \theta \leq 2\pi$, <u>and let</u> $\lim\limits_{\delta\to 0} \omega(\delta) \log \frac{1}{\delta} = 0$. <u>Let</u> $\Pi_n(x)$ <u>be the polynomial of degree</u> $\leq 3n-1$ <u>which interpolates</u> $f(z)$ <u>in the</u> n^{th} <u>roots of unity</u> $\{z_k\}_1^n$, <u>and</u> $\Pi_n^{(n-1)}(z_k) = \beta_{kn}$, $\Pi_n^{(n)}(z_k) = \gamma_{kn}$, <u>where</u>

$$(2.25) \qquad \beta_{kn} = o\left(\frac{n!}{n^3 \log n}\right) \quad , \quad \gamma_{kn} = o\left(\frac{n!}{n^2 \log n}\right) \quad .$$

<u>Then</u> $\Pi_n(z)$ <u>converges to</u> $f(z)$ <u>uniformly in</u> $|z| \leq 1$.

3. <u>Proof of Theorem 1</u>.

We shall only show how to obtain (2.12). The proof for (2.13) and (2.14) is similar and is omitted. Set $R_k(x) = r_k(x) + a_k Q(x)$ where a_k is a constant and $r_k(x)$ is a polynomial of degree $\leq n-2$. Then $R_k(x)$ already satisfies the conditions $R_k^{(n-1)}(x_j) = R_k^{(n)}(x_j) = 0$, $j = 1,\ldots,n$ which are the last two conditions in (2.9). In order to have $R_k(x_j) = \delta_{kj}$ $(j = 1,\ldots,n)$, the polynomial $r_k(x)$ must satisfy the conditions

$$(3.1) \qquad r_k(x_j) = \begin{cases} -a_k Q(x_j) & , \quad j \neq k \\ 1 - a_k Q(x_k) & , \quad j = k \quad . \end{cases}$$

Since $r_k(x)$ is a polynomial of degree $\leq n-2$, we then have

$$(3.2) \qquad r_k(x) = -a_k \sum_{j=1}^{n} Q(x_j) \frac{w(x)(x_j-x_k)}{(x-x_j)(x-x_k)w'(x_j)}$$

$$= -a_k\{L_n(Q;x) - \frac{w(x)}{x-x_k}[x_1,\ldots,x_n;Q]\} .$$

Condition (3.1) for $j = k$ yields, using Lemma 2.1, $\frac{1}{a_k} = w'(x_k)[x_1,\ldots,x_n;Q]$ which combined with (3.2) gives (2.12). This completes the proof of Theorem 1. Theorem 2 is now easy to prove on putting z_k for x_k which is permissible because of (2.17). Also, observe that when $z_j = e^{2j\pi i/n}$ $(j = 1,2,\ldots,n)$ then

$$(3.3) \qquad \ell_k(z_j t) = \frac{(t^n-1)z_k}{(tz_j-z_k)n}$$

$$= \begin{cases} \ell_{k-j}(t) & , \quad j < k \\ \ell_{n+k-j}(t) & , \quad j \geq k \end{cases} .$$

We also use the identity (which is an immediate consequence of Hermite interpolation formula)

$$(3.4) \qquad \sum_{j=1}^{n} \ell_j^2(t)[1 - \frac{n-1}{z_j}(t-z_j)] = 1 ,$$

to verify that $[z_1,\ldots,z_n;\Omega_k] = \frac{1}{n!}$, $(k = 1,\ldots,n)$.

Furthermore,

$$[z_1,\ldots,z_n;\tau_k] = \sum_{j=1}^{n} \frac{\tau_k(z_j)}{w'(z_j)}$$

$$= \frac{n-1}{n!} \sum_{j=1}^{n} \int_0^1 (tz_j-z_k)\ell_k^2(tz_j)(1-t)^{n-2}dt$$

so that on using (3.3) we easily obtain

$$[z_1,\ldots,z_n;\tau_k] = \frac{z_k}{n^2(n-2)!}\int_0^1 (t^n-1)(1-t)^{n-2}dt \quad .$$

The formulae (2.22), (2.23) and (2.24) are now easy to deduce from (2.12), (2.13) and (2.14) respectively on using (2.17), (2.18) and (2.19). We omit the details.

4. Estimates on the Fundamental Polynomials of Theorem 2. Proofs of Theorems 3 and 4.

We shall now obtain some estimates for the fundamental polynomials of Theorem 2. We have

Lemma 4.1. *For $|z| \le 1$, we have the following estimates for polynomials $R_k(z)$, $S_k(z)$, $T_k(z)$ of Theorem 2, for $n = 2,3,\ldots$*

$$\sum_{k=1}^{n} |R_k(z)| \le 16 + \log n \tag{4.1}$$

$$\sum_{k=1}^{n} |S_k(z)| = O\left(\frac{n^3 \log n}{n!}\right) \tag{4.2}$$

$$\sum_{k=1}^{n} |T_k(z)| = O\left(\frac{n^2 \log n}{n!}\right) \quad . \tag{4.3}$$

Proof. Since for $|z| < 1$,

$$\left|\int_0^1 (1-z^n t^n)^2 (1-t)^{n-2}dt\right| \le 4\int_0^1 (1-t)^{n-2}dt = \frac{4}{n-1}$$

and

$$\left|\int_0^1 (1-t^n)^2 (1-t)^{n-2} dt\right| \geq \int_0^1 (1-t)^n dt = \frac{1}{n+1}$$

we have from (2.19) and (2.17), and $n = 2,3,\ldots$

(4.4) $$|W(z)| \leq 4\left(\frac{n+1}{n-1}\right) + 1 < 13$$

so that (2.22) yields

(4.5) $$|R_k(z)| \leq |\ell_k(z)| + \frac{13}{n} \quad .$$

On using the known inequality (see [15]),

(4.6) $$\sum_{k=1}^{n} |\ell_k(z)| \leq 3 + \log n \quad , \quad |z| \leq 1 \quad ,$$

we get (4.1) from (4.5).

In order to prove (4.2), we observe that $|\ell_k(zt)| \leq 1$ for $|z| \leq 1$ $(0 \leq t \leq 1)$ so that from (2.20) we have for $|z| \leq 1$,

$$|\Omega_k(z)| \leq \frac{1}{(n-2)!} \int_0^1 \Big[1+(n-1)(|zt|+|z_k|)\Big](1-t)^{n-2} dt < \frac{2n}{(n-1)!}$$

so that

(4.7) $$\left|\Omega_k(z) - \frac{\Omega_{\lambda(j,k)}(1)}{z_j}\right| \leq \frac{4n}{(n-1)!} \qquad (j,k=1,\ldots,n) \ .$$

Then from (2.23) on using (4.4), (4.6) and (4.7) we have

$$|S_k(z)| \leq \frac{4n(3 + \log n)}{(n-1)!} + \frac{13}{n!}$$

which at once gives (4.2).

The proof of (4.3) now follows similarly from (2.21), (2.24), (4.4) and (4.6).

<u>Lemma 4.2</u>. (Kiš [15]). <u>If</u> $f(z)$ <u>is analytic in</u> $|z| < 1$ <u>and continuous in</u> $|z| \le 1$, <u>and if</u> $F_n(z)$ <u>is the Jackson mean, then</u>

$$|f(e^{i\theta}) - F_n(e^{i\theta})| \le 6\omega\left(\frac{1}{n}\right)$$

<u>where</u> $\omega(\delta)$ <u>is the modulus of continuity of</u> $f(z)$. <u>The explicit form of Jackson mean</u> $F_n(z)$ <u>of degree</u> $2n-2$ <u>is given by</u>

$$F_n(z) = \frac{3}{(2n^2+1)2\pi ni} z^{2-2n} \int_{|t|=1} f(t)t^{1-2n}\left(\frac{t^n - z^n}{t-z}\right)^4 dt \quad .$$

<u>Proof of Theorem 3</u>. Let $N = [n/2]$. Then $F_N(z)$ is a polynomial of degree $\le n-2$ and so $F_N^{(p)}(z)$ vanishes identically for $p \ge n-1$. Therefore

$$\begin{aligned} f(z) - \Pi_n(z) &= f(z) - F_N(z) + F_N(z) - \Pi_n(z) \\ &= f(z) - F_n(z) + \sum_{k=1}^{n} \{F_N(z_k) - f(z_k)\}R_k(z) \\ &\quad - \sum_{k=1}^{n} \beta_k S_k(z) - \sum_{k=1}^{n} \gamma_k T_k(z) \quad . \end{aligned}$$

Using Lemma 4.1, and (2.25) we have

$$|f(z) - \Pi_n(z)| \le 6\omega\left(\frac{1}{N}\right) + 6\omega\left(\frac{1}{N}\right)(16 + \log n) + o(1) = o(1)$$

since $\omega(\delta)\log\frac{1}{\delta} \to 0$ as $\delta \to 0$. This completes the proof of Theorem 3.

<u>Remark</u>. To avoid any misunderstanding, we set

$$\Pi_n(z) \equiv \Pi_n(f,z) \quad .$$

Theorem 3 gave sufficient conditions on the function $f(z)$ for the convergence of the $(0,n-1,n)$ interpolating polynomial $\Pi_n(f,z)$ to converge uniformly to $f(z)$ in $|z| \le 1$. The Dini-Lipschitz condition imposed on $f(z)$ in Theorem 3 is necessary as the following theorem shows.

Theorem 4. There exists a function $f(z)$ analytic in $|z| < 1$ and continuous in the closed disk $|z| \le 1$ such that the sequence $\{\Pi_n(f,z)\}_{n=1}^{\infty}$ of $(0,n-1,n)$ interpolating polynomials for the equidistant interpolating nodes

$$(4.8) \qquad \omega_k = e^{(2k-1)\pi i/n} \qquad (k=1,2,\ldots,n)$$

with the $2n$ additional conditions

$$(4.9) \qquad \Pi_n^{(n-1)}(\omega_k) = \Pi_n^{(n)}(\omega_k) = 0 \qquad (k=1,2,\ldots,n)$$

diverges at $z = 1$. Indeed we have $\overline{\lim_{n\to\infty}} \Pi_n(f,1) = \infty$.

Proof. Set $W^*(z) = \prod_{j=1}^{n} (z-\omega_j)$ where ω_j is defined by (4.8). Then from (2.1), (2.2) and (2.5), with $x_k = \omega_k$, we easily have

$$(4.10) \qquad W^*(z) = z^{n-1}\left\{\frac{\int_0^1 (1+z^n t^n)^2 (1-t)^{n-2}dt}{\int_0^1 (1-t^n)^2 (1-t)^{n-2}dt} - 1\right\}.$$

Setting $z = 1$ in (4.10) it easily follows that

$$(4.11) \qquad 0 \le W^*(1) \le 13 \qquad (n=2,3,\ldots).$$

Then from (2.12) and (4.9) we have

$$(4.12) \qquad \Pi_n(f,z) = \sum_{k=1}^{n} f(\omega_k)\left[\ell_k^*(z) + \frac{W^*(z)}{w^{*\,'}(\omega_k)}\right] .$$

Now consider the polynomials

$$(4.13) \quad P_{2n}(x) = \frac{1}{n} + \frac{z}{n-1} + \frac{z^2}{n-2} + \ldots + \frac{z^{n-1}}{1} - \frac{z^{n+1}}{1} - $$

$$- \frac{z^{n+2}}{2} - \ldots - \frac{z^{2n}}{n} \qquad (n=1,2,\ldots) .$$

Fejér (see [28], p. 92) has shown that

$|P_{2n}(e^{i\theta})| \le 2\lambda$ where $\lambda = \int_0^\infty \frac{\sin x}{x}\,dx$.

Thus, all polynomials $P_{2n}(z)$ $(n = 1,2,\ldots)$ are bounded on $|z| = 1$, hence for $|z| \le 1$, by 2λ . We have

$$(4.14) \qquad \Pi_n(P_{2n},z) = \sum_{k=1}^{n} P_{2n}(\omega_k)\left[\ell_k^*(z) + \frac{W^*(z)}{w^{*\,'}(\omega_k)}\right]$$

$$= L_n(P_{2n},z) - \frac{W^*(z)}{n} \sum_{k=1}^{n} \omega_k P_{2n}(\omega_k)$$

where $L_n(P_{2n},z)$ is the Lagrange interpolation polynomial of degree $n-1$ for $P_{2n}(z)$ with nodes $\omega_k = e^{(2k-1)\pi i/n}$ $(k=1,2,\ldots,n)$. Since $\sum_{k=1}^{n} \omega_k P_{2n}(\omega_k) = -\frac{n^2}{n-1}$, we have

$$(4.15) \qquad \Pi_n(P_{2n},z) = L_n(P_{2n},z) + \left(\frac{n}{n-1}\right)W^*(z)$$

so that from (4.11) we get

$$(4.16) \qquad \left(\frac{n}{n-1}\right)W^*(1) \ge 0 \qquad (n=1,2,\ldots) .$$

Now, from a result of Fejér (see [28], p. 92) we have

(4.17) $$L_n(P_{2n},1) > 2\ell og\ n$$

so from (4.15), (4.16) and (4.17) we have $\Pi_n(P_{2n},1) > 2\ell og\ n$.

The remainder of the proof of Theorem 4 follows by an argument identical to that given by Fejér (see [28], p. 92) for the Lagrange interpolation polynomials.

5. Least Squares Convergence.

Let $z_k = e^{2k\pi i/n}$. We shall prove the following.

Theorem 5. *Let $f(z)$ be analytic in $|z| < 1$ and continuous in $|z| \leq 1$. Let $\Pi_n(z)$ be the polynomial of degree $3n-1$ coinciding with $f(z)$ in the n^{th} roots of unity and with the $2n$ additional conditions*

$$\Pi_n^{(n-1)}(z_k) = \Pi_n^{(n)}(z_k) = 0 \quad (k=1,2,\ldots,n)$$

Then the sequence $\Pi_n(z)$ converges to $f(z)$ on $|z| = 1$ in the mean of second order. Consequently

(5.1) $$\lim_{n\to\infty} \Pi_n(z) = f(z)$$

uniformly in $|z| \leq r < 1$.

Proof. Let

(5.2) $$I_n = \int_{|z|=1} |f(z) - \Pi_n(z)|^2 |dz|$$

where $\Pi_n(z) = \sum_{k=1}^{n} f(z_k)R_k(z)$ and $R_k(z)$ is as defined

in (2.22); $\ell_k(z) = \frac{z_k}{n}\left(\frac{z^n - 1}{z - z_k}\right)$. Further, let

$\Delta_n(z) = f(z) - t_{n-2}(z)$ and $E_n = \max\limits_{|z|=1} |\Delta_n(z)|$ where

$t_{n-2}(z)$ is the polynomial of degree $n-2$ of best Tchebycheff approximation to $f(z)$ on C . It is our object to show that

$$\lim_{n\to\infty} I_n = 0 \quad . \tag{5.3}$$

Now, let C denote the circle $|z| = 1$.

$$I_n = \int_C |f(z) - t_{n-2}(z) + t_{n-2}(z) - \Pi_n(z)|^2 |dz|$$

$$\leq 2\int_C |f(z) - t_{n-2}(z)|^2 |dz| + 2\int_C |t_{n-2}(z) - \Pi_n(z)|^2 |dz|$$

$$= I_n' + I_n'' \quad .$$

Now

$$I_n' = 2\int_C |\Delta_n(z)|^2 |dz| \leq 4\pi E_n^2$$

$$I_n'' = 2\int_C \left|\sum_{k=1}^{n} \left[t_{n-2}(z_k) - f(z_k)\right] R_k(z)\right|^2 |dz|$$

$$\leq 2 \sum_{k=1}^{n} \sum_{j=1}^{n} |\Delta_n(z_k)\overline{\Delta_n(z_j)}| \left|\int_C R_k(z)\ \overline{R_j(z)}|dz|\right|$$

$$\leq 2(E_n)^2 \sum_{k=1}^{n} \sum_{j=1}^{n} \Lambda_{j,k}$$

where

$$\Lambda_{j,k} = \left|\int_C \ell_k(z)\,\overline{\ell_j(z)}\,|dz|\right| + \left|\frac{z_k}{n}\int_C \overline{\ell_j(z)}\,W(z)\,|dz|\right|$$

$$+ \left|\frac{\overline{z}_j}{n}\int_C \ell_k(z)\,\overline{W(z)}\,|dz|\right| + \left|\frac{z_k z_j}{n^2}\int_C |W(z)|^2\,|dz|\right|$$

$$= \Lambda_{j,k}^{(1)} + \Lambda_{j,k}^{(2)} + \Lambda_{j,k}^{(3)} + \Lambda_{j,k}^{(4)} \quad .$$

Since C denotes the circle $|z| = 1$, we have, using (2.16)

$$\int_C \ell_k(z)\overline{\ell_j(z)}\,|dz| = \frac{2\pi z_k \overline{z}_j}{n^2}[1 + z_k\overline{z}_j + (z_k\overline{z}_j)^2 + \ldots + (z_k\overline{z}_j)^{n-1}]$$

$$= \frac{2\pi\delta_{kj}}{n} \qquad (\delta_{kj} = \text{Kronecker delta}) \; .$$

Thus $\Lambda_{j,k}^{(1)} = \frac{2\pi\delta_{kj}}{n}$. Let $\frac{1}{D_n} = \int_0^1 (1-t^n)^2 (1-t)^{n-2}dt$. Then from (2.17) $W(z) = A_n + B_n z^n + C_n z^{2n}$ where for $n=2,3,\ldots$

$$\text{(5.4)} \quad \begin{cases} A_n = D_n\left[\int_0^1 (1-t)^{n-2}dt - 1\right] \leq 4 \\ B_n = -2D_n\int_0^1 t^n(1-t)^{n-2}dt \;\; ; \quad |B_n| \leq 6 \\ C_n = D_n\int_0^1 t^{2n}(1-t)^{n-2}dt \leq 3 \quad . \end{cases}$$

Then

$$\overline{\ell_j(z)}\;W(z) = \frac{\overline{z}_j}{n}[z^{-(n-1)} + \overline{z}_j z^{-(n-2)} + \ldots + (\overline{z}_j)^{n-1}]W(z)$$

$$= \frac{1}{n}[A_n + \text{terms in } z^k \; (k \neq 0)] \quad .$$

Thus, using (5.4) we have

$$\Lambda^{(2)}_{j,k} = \left|\frac{z_k}{n}\int_C \overline{\ell_j(z)}\ W(z)\,|dz|\right|$$

$$\leq \frac{2\pi}{n^2} A_n \leq \frac{8\pi}{n^2} \qquad (n=1,2,\ldots) .$$

By an identical computation we have $\Lambda^{(3)}_{j,k} \leq \frac{8\pi}{n^2}$. Again using (5.4) we have

$$\Lambda^{(4)}_{j,k} = \left|\int_C \frac{z_k \overline{z}_j}{n^2}\ W(z)\overline{W(z)}\,|dz|\right| \leq \frac{2\pi}{n^2}\ 16 + 36 + 9) = \frac{122\pi}{n^2} .$$

Therefore

$$I''_n \leq 2(E_n)^2 \sum_{k=1}^{n} \sum_{j=1}^{n} \left|\frac{2\pi\delta_{kj}}{n} + \frac{8\pi}{n^2} + \frac{8\pi}{n^2} + \frac{122\pi}{n^2}\right|$$

$$= 2(E_n)^2 \sum_{k=1}^{n} \sum_{j=1}^{n} \left|\frac{2\pi\delta_{kj}}{n} + \frac{138\pi}{n^2}\right| = 280\pi(E_n)^2 .$$

So, $I_n = I'_n + I''_n \leq 4\pi(E_n)^2 + 280\pi(E_n)^2 = 284\pi(E_n)^2$. From [32] (Theorem 5, p. 36) $E_n \to 0$ as $n \to \infty$. Therefore (5.3) holds. By the Cauchy integral formula

$$(5.5) \qquad [f(z) - \Pi_n(z)]^2 = \frac{1}{2\pi i}\int_C \frac{[f(t) - \Pi_n(t)]^2}{t-z}\,dt$$

(5.1) easily follows from (5.5). This proves the theorem.

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